Teachers' Inquiry in
Mathematics Education

TIMEless

A short introduction to Lesson Study – TIMEless ideas for professional development

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A short introduction to Lesson Study – TIMEless ideas for professional development

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Table of Contents

Introduction

This document is meant to serve as an in-depth introduction to Lesson Study and should help to ensure that all participants in the TIME project share basic knowledge from the literature and also some practical principles for the implementation of Lesson Study in the partner schools. It can be considered as an elaboration of the comments provided in the *Template for lesson plans* (TIMEplate), adding details and justifying the basic advice given there. The text also lists some theoretical and practical reasons why Lesson Study is a good candidate to support teachers' inquiry on mathematics education.

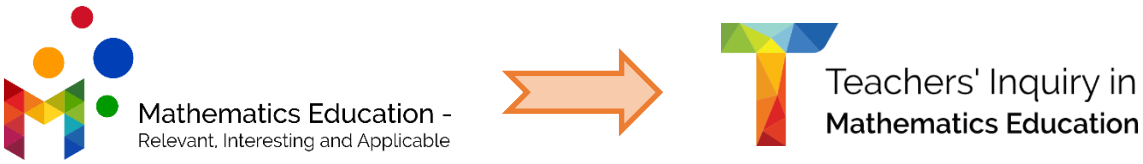
Today, many teachers have numerous and mixed experiences with professional development courses and activities, whether these are proposed to introduce new fields of knowledge considered relevant in the context of a new curriculum, or they strive to introduce new pedagogical or didactic ideas that are thought to improve students' learning and motivation. Such activities can be organised as lectures, seminars, or talks at universities or local schools; in these activities and courses the teacher takes on the role of a student.

The format for professional development proposed in the TIME project is somewhat different. We aim to introduce teachers to the culture of Lesson Study, which is a teacher-led form of life-long professional development. Lesson Study works as an integrated part of the professional life of most Japanese teachers in primary and lower secondary schools. It has many variations in Japan and certainly, adapting it to upper secondary school in other countries is not a straightforward copy-paste job. Over the past 30 years, Lesson Study has been implemented in many other countries and is considered by many professionals as having strong potential for developing the teachers' practice and further their students' learning.

This document addresses the questions teachers often pose when encountering this practice for the first time:

- What is it?
- How do we do it?
- Are certain elements required?
- Why do we do it?
- How do we get started?
- What are the "rules"?
- How often do we do it?

The answers to these questions are based on practical advice and supported by knowledge from the research literature, which draws on theories presented and shared in the **MERIA project** (https://meria-project.eu/). This will further be linked to the methodology and templates for teachers' inquiry.



We also devote a section to the task of "leading Lesson Study" in the sense, that it is a special challenge to engage teachers in Lesson Study, ensuring that the planning and reflection go beyond just sharing of best practices. Teachers are more used to sharing practices and have no experience with discussions of who among the students learnt what in the context through which they were taught. This will be the learning goal for teachers in the TIME project; to engage in reflections on students' learning to adjust their teaching practice to further it.

Chapter 1. Why Lesson Study?

Imagining the future of teacher inquiry

We present a story of three teachers being involved in a teaching situation which might be very familiar to many teachers around the world. It shows the need for teachers to discuss their practice, but at the same time, the lack of time and framework to organize their collaboration. In the story, teachers make decisions on the go and base them on their experience. Hopefully, the story will motivate the need for directions coming from Lesson Study, which are presented in the rest of this booklet.

One day a team of upper secondary mathematics teachers got involved in a discussion during the lunch break.

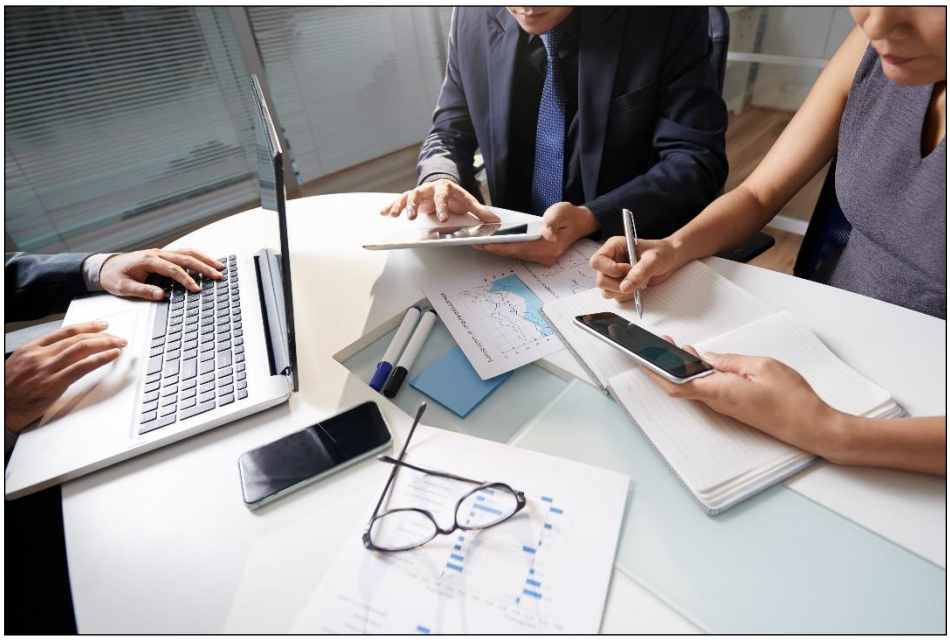
- *Anna: I'm about to introduce the exponential functions in grade 10, but I doubt that I should keep on doing this in the same way as before.*
- *Ben: What do you mean?*
- *Anna: Well, usually the exponential function is just presented by the formula and successful students quickly get to accept it. They can manipulate and use the formula in different contexts, where it is again presented just as a model of a certain situation. Then students, more or less successfully, answer questions by plugging given variables into the formula. I think it is really boring. It brings students no idea of the properties of the function. Nor do students develop any appreciation of the meaning of the exponent x, which is meant to be any real number, or how it evolves from integer exponents.*
- *Clara: Right, for my students, calculation by exponents involves just a button on the calculator. The textbook proposes no enigma on these calculations! We just simply begin to operate with real exponents without any other explanation and we do it by a calculator. I would really like to have an opportunity to try with my students…*

Bell rings and the third teacher barely finishes the sentence: "…a different approach, and to consider this a little bit more deeply." The teachers run to their classrooms and the conversation seems to stop at that moment.

This might be one of many similar situations in which teachers feel the need to discuss their subject with their colleagues, but they find it hard to continue the conversation by meeting again on top of their preparation and teaching duties. Introducing a new mathematical concept in high school is always challenging – each generation of students is different, and the textbooks provide only an outline for the lesson. The teachers need to adapt to current students and draw heavily on their experience, but at the same time, they feel time-pressured and uncertain on how to proceed. Similar situations have been evidenced in project MERIA (Bašić, Milin Šipuš, 2019). During the interviews, teachers expressed that they lack rich resources for innovative teaching and time to come up with new teaching ideas. Also, they would like to use materials they

trust, which comes from knowing the authors or by experiencing new ideas in a lesson during a workshop.

The three teachers pass next to each other down the hall after the lesson. Still considering this teaching situation, Anna stops Ben and Clara and proposes that they work together on a new approach to introduce students to the exponential function. To save time, they agree to collect some teaching proposals and resources and bring them to their meeting next week. They will all check different textbooks, ministerial teaching guides, and teaching material, such as worksheets received from their colleagues, in some cases even several years ago. They will also search the internet and the school library. One teacher even finds his textbooks from the university to recall how the topic was addressed during his studies. But no textbook really deals with the concept a^x, for a general $x \in \mathbb{R}$, nor does it explain whether any assumptions should be made for the basis a.



When the teachers meet again, they discuss the different approaches they found in the literature. First, they discuss the issue of dealing with the vast possibilities of real numbers.

- *Ben: Well, we all explain the meaning of a^n and $a^{m/n} = \sqrt[n]{a^m}$ for $m, n \in \mathbb{N}$. Now, the next step is important. We must rely on an informal notion of real numbers.*
- *Clara: That's true, but I never experienced students having difficulties accepting real numbers.*
- *Anna: Sure, they don't question the existence of real numbers, but sometimes I noticed that students struggle with infinite decimal numbers. For some of them, π remains 3.14 for all their life!*
- *Clara: Hm, perhaps we could work on this by illustrating the way real numbers are in general approximated by numbers with finitely many decimals. Then we could let students approximate a^x similarly and support this by experiments using a calculator.*

- *Anna: I am only reluctant to do everything in a very abstract way. The students might get confused with what we are actually doing. I think I will take a concrete example of calculating 2^{π}. This way, I'll discuss the number π, too.*
- *Clara: I'll do that also with my class, but I think my students should also recall the fractional exponents, so I will let them investigate what $2^{1/2}$ and $2^{\sqrt{2}}$ mean.*
- *Ben: I'm not sure I want to work just with concrete examples. I think I will still try to have the students deal with a more general abstract way. Next week we can compare our lessons!*
- *Anna: Ok, let's try out our different strategies!*

This discussion shows that the topic of introducing real exponents to define the exponential function is very rich and that the teacher faces many decisions in preparing such a lesson. Only in terms of mathematics, different teachers may emphasize different general goals: emphasising the nature of real numbers by approximating them with finite decimals, recalling the definition of powers with integer and rational exponents, working on students' mathematical literacy and precision in defining concepts, or developing students' inquiry skills in dealing with non-routine problems.

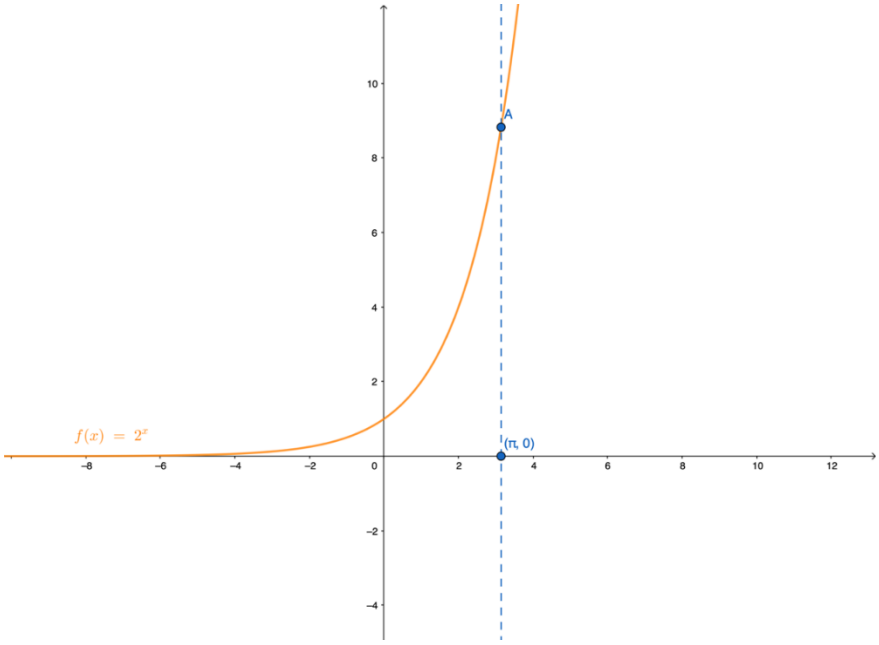


Figure 1. The graph of the exponential function with base 2 and the point $(\pi, 2^{\pi})$.

To prepare for the lesson, the teacher would write a lesson plan containing a description of the time management, how to introduce the problem of defining the expression a^x for a general $x \in \mathbb{R}$ (including incorrect and naive definitions that students might come up with when discussing the topic for the first time), students' prerequisites, organisation of the students' work, concrete examples and worksheets to use and possible ways to support students' discussion. There are still many more details to complete the lesson, just to mention a few: the use of blackboard and technology, the dynamics of the lesson, etc.

After the teachers delivered their lesson, they met to discuss. First, they reported some of the most interesting moments, as they recalled them.

- *Clara: As we agreed, the students worked in groups for 30 minutes. They were allowed to use calculators, their textbooks, and the internet. After that, I conducted a whole class discussion on given answers and proceeded to explain the general definitions outlined in the discussion, while using $2^{1/2}$ and $2^{\sqrt{2}}$ as examples. Actually, I have the impression that 30 minutes was too much time and that some of the students did not know how to proceed.*
- *Ben: I agree, we could have adjusted the lesson to allow only 15 minutes for the initial group work, to reserve most of the lesson for collecting answers from the groups and in the end to exemplify the more general definitions. We usually spend no time on this problem, so spending a full lesson to raise it and try to explain the answer is a big add-on.*
- *Anna: I used π as the exponent and this really opened the discussion in a direction that I did not expect. In the end, we spent more time discussing real numbers and infinitely many decimals of π, than working on the exponential function. So, I am more interested in how the students dealt with $2^{1/2}$and $2^{\sqrt{2}}$.*
- *Clara: Well, the students immediately knew that $2^{1/2} = \sqrt{2}$, but I was surprised at how many different answers students gave to the second question, including*
  - "$2^{\sqrt{2}}$ is 2 multiplied by itself about 1.4 times"
  - " $2^{\sqrt{2}}$ is close to $2^{1.41}$, but adding more decimals is more precise" (demonstrated with the values of $2^{1.414}$ and $2^{1.41421}$; students note that "we always need to add a bit more")
  - "$\sqrt{2}^2 = 2 \rightarrow 2^{\sqrt{2}^2} = 4 \rightarrow 2^{\sqrt{2}} = 2$."

  These different proposals were written on the blackboard next to each other by the students themselves and subsequently explained orally to the whole class.
- *Anna: Yes, I have seen this mistake in using the exponent rule, such as in the third "answer" above, many times. It may be better to discuss that in public.*
- *Ben: Do you have any feedback - did the students understand the lesson?*
- *Clara: Well, there was not enough time. I will need to discuss that next time. Some students seriously tried to define $2^{\sqrt{2}}$, but generally used little other sources than their CAS tool. So, I am questioning the usefulness of spending 20 minutes on all the "confusion" that the resulting answers represent. After the discussion of these different ideas, I proceeded to explain the "official definitions", first in the case of 2^x and then briefly claimed, "the same can be done for other positive numbers in place of 2". All in all, I think it went well.*

This dialogue does not say much about the outcome of the lessons, but we imagine that a similar conversation could be held by a group of teachers at any school. The idea to change the teaching activity or to compare experience is not at all something new to any practising teacher, but we will raise questions about the way to perform similar teachers' inquiries. Showing different, and in part erroneous, students' ideas could be

good preparation for the lecture-type presentation, and also keeping the students' "answers" visible – rather than erasing them, as Clara did – would have been even better, to compare them to the "textbook definition".

In the story, the teachers are retelling what was happening during the class from their memory, but it is natural to be selective or to forget something. Also, in a classroom with many active students, it is hard for the teacher to follow all the students and make sure that all important ideas (including the wrong ones) get attended to by the class as a whole, in particular, to eliminate misconceptions not by the teachers' authority but by rational arguments, ideally produced by students.

One aspect of this story is that it might be beneficial for teachers to observe each other's lesson to make the discussion afterwards more informative. The discussion is better if the teachers participate in the class than if they are just told what happened. The observing teacher can notice something that the teacher who teaches the lesson cannot, e.g., that the students have been thinking about something beyond the use of a calculator, which may point to their productive ideas.

Another aspect of this story is that each teacher chose to do the lesson in a slightly different manner. This might show the advantages and drawbacks of different approaches but makes it harder for the teachers to compare their experience. In this way, the group of teachers did not focus on just one single goal of the lesson and in the end, they could not determine to which level their experiment was successful. Nonetheless, the experience might be very valuable for the involved teachers – both to understand by which teaching activity the goal was realized and to reflect on their style of teaching.

If the group of teachers is even more ambitious, they might organize more than one implementation of the lesson, each time with a different teacher and an improved lesson plan. Again, the group could observe the lesson and have a shared meeting afterwards. They could discuss the changes to the lesson plan, the effects they had on the outcome of the lesson, and what changed due to a different group of students. They might also determine what the students and the teachers learned from the proposed plan of the lesson and discuss the possibility of sharing their design and observations with other mathematics teachers at their school.

This story of the reflection of three teachers about a teaching problem is an imaginary one. It could, in principle, take place in a context where a study of a lesson is part of the culture surrounding the teacher profession, as it is in Japan. Lesson Study can take on many forms, it can be carried out by a novice teacher who is observed by more experienced colleagues, who afterwards help him reflect on his teaching, or as a study organized by a team of experienced teachers developing a teaching scenario (lesson plan) with repeated observations as mentioned above. It could be a closer collaboration

of a few teachers, where they plan together everything in detail, it could be a collaboration of all mathematics teachers in one school working as described above, or it could be a collaboration also with teachers from other disciplines and representatives from the management taking part in some of the planning meetings, observations, and reflections.

In any case, variations are many, but a group of teachers eager enough to take on such an endeavour might need guidance in the form of principles that will somewhat ensure that the experience is worth the time and the effort. In the following sections and chapters, we will describe the sources and basic principles of Lesson Study that have been used to improve teaching practice in Japan for almost 150 years.

Questions for reflection:

- What is the definition of an exponential function that you use at your school?
- Recall when you discussed a lesson idea with a colleague multiple times, before and after the actual lesson.
    - What did you learn from this?
    - How could you improve this practice?
    - How could you make this happen more structurally?

The historic background of the Lesson Study

In Japan, Lesson Study has been widely used for over a century (Makinae, 2010) and is the primary method of professional development for teachers ever since public education became an institution (Lewis and Tsuchida, 1998). It grew out of the transformation of the Japanese educational system from individualized to group instructions. Individualized instructions assumed teaching students individually, according to their abilities, and were provided to common people in the *temple schools (terakoya)* until the late 19th century. With the development of commerce and the collapse of the class system, in 1872 the Meiji government established *teachers' schools* and foreign teachers were invited to Japan to disseminate Western scholarship (Isoda, Stephens, Ohara & Miyakawa, 2007). Western teachers introduced the concept of a whole classroom instruction, which was still rare also in the West. This interaction has led to writing new textbooks and the emergence of open lessons[1] used by teachers to discuss and adapt their teaching practice. Supported by the government, Japanese teachers throughout the country adopted the culture of conducting class observations and holding critique sessions. Teachers played the central role in making new approaches practical and understandable, and in this way, Lesson Study worked effectively to connect theory and practice.

[1] Lessons that are made public for a wider audience, e.g., teachers from other schools, researchers, etc.

Lesson Study was (and still is) often used to study and understand new educational approaches and implement a new national curriculum. In the process of doing Lesson Study, Japanese teachers began proposing new teaching methods with a focus on problem solving. In this process, they changed their teaching from teacher-led lecture lessons to student-centered problem-solving lessons where students could pose questions and discuss with one another. Today problem-solving is well known as a major way of teaching mathematics in Japan. It is characterized by the "open approach" – promoting the use of various ways of solving problems, posing problems with many answers, and performing activities of changing and developing problems by the students.

Figure 2. Japanese teachers observe a research lesson at Oshihara Elementary School in Japan as part of Lesson Study, June 2012. (Source: www.apmreports.org, Photo: Tom McDougal)

Lesson Study in Japan comes in different shapes and sizes: small school-based Lesson Study with a single team of teachers as well as large-scale, national-level Lesson Study where teachers often travel long distances to participate, and hundreds of teachers, teacher educators, and researchers gather for one event.

Lesson Study first came to the attention of educators and teachers outside Japan primarily through the publication of The Teaching Gap (Stigler and Hiebert, 1999) and the TIMMS video study, which showed the structure of Japanese mathematics teaching in the classroom. Independently, some educators (Lewis and Tsuchida, 1998) noticed the significance of Japanese Lesson Study and brought this to the attention of the international community. Since then, many mathematics teachers, researchers, and teacher educators around the world have been involved in Lesson Study. Lesson Study has subsequently spread to many classrooms around the world, but also to a dozen of international conferences and workshops in which teachers, teacher educators, and researchers share their experience and progress with Lesson Study.

Questions for reflection:
- Is there any tradition or infrastructure for Lesson Study in your country?
- Can you find information or videos about Japanese Lesson Study online?

Why do Lesson Study?

It is a common experience of many mathematics teachers who have attended in-service courses on mathematical modelling, inquiry-based teaching, and so on, that besides possibly trying out the teaching materials which are handed out during such events, little is changed once the teacher returns to his day-to-day teaching. If the course requires that the participants develop their materials during the course, these might be used as well, but it is not likely that the teachers keep developing more materials or in general keep changing their teaching according to the proposals they meet at the in-service course. This was the experience of the EU funded PRIMAS project, where Garcia (2013) proposed that structures inspired by Lesson Study might support more sustainable implementation of inquiry-based teaching as a more permanent approach for teachers when planning their lessons.

This was to some extent tried within a research project (Jessen, in preparation) with an online course on inquiry-based mathematics teaching, where the participants met several times during 5 weeks (expected workload was 80 hours), to learn how to design inquiry-based teaching. The participants engaged in a detailed study and analysis of a self-selected teaching problem, designed a lesson, described it in a lesson plan, and (in most cases) realised it in their classrooms before sharing it with the other participants and educators through an online platform.

Even very well taught courses that succeed in engaging the teachers in new ideas and provide them with well-designed lesson plans may fail to substantially affect the way the teachers teach, especially in the short term. The Mist Project in the USA has studied the question "What does it take to support improvements in the quality of teaching (and thus student learning) on a large scale?" (Cobb et al., 2018), where they followed the teachers of 360.000 students (2007-2011), and later included the school managements in a similar study including 180.000 students (2011-2015). **They concluded that when it comes to the development of teaching practice, Lesson Study is the most promising way to ensure continuous professional development.**

Papers and books written by Japanese teachers and researchers are very important sources to provide us with insights about Lesson Study in practice. Isoda, Stephens, Ohara and Miyakawa (2007) formulate the purpose of Lesson Study as:

> Underlying the practice of Lesson Study is the idea that teachers can best learn from and improve their practice by seeing other teachers teach. Second, there is an expectation that teachers who have developed deep understanding of a skill in subject matter pedagogy should be encouraged to share their knowledge and experience with colleagues. Thirdly, while the focus appears to be on the teacher, the final focus is on the cultivation of students' interest and on the quality of their learning. The various cycles of refinement which are at the heart of the Japanese Lesson Study only make sense in terms of improving the quality of the students' learning. (p. xvi).

There is a striking similarity between Lesson Study as an activity for teachers, and the experience in inquiry-based education aimed at the students: namely, the principle that people learn from studying a problem through experimenting with hypothetical solutions.

For teachers, problems are related to students' learning (with specific and more general goals), and they keep fine-tuning the experiment until they are ready to share their findings with others. Furthermore, it is noted that "Lesson Study does not refer just to in-school training (or in our words, simply to observing another teacher's lesson). It is the process by which the teachers of mathematics at several schools in the same community work together to research teaching materials, develop teaching plans (lesson plans) and practice teaching lessons" (Isoda, Stephens, Ohara & Miyakawa, 2007). The authors add that it is a common practice in Japan to secure the conditions for the teachers to use part of their week (around two hours) for such activities, but some also invest their private time now and then. In the same book, authors from other countries in North America, Asia, and Africa, report on their efforts to implement Lesson Study in environments where it is not common; they report on issues related to time and resources, as well as other constraints and conditions for implementing Lesson Study outside of Japan.

Introducing secondary level mathematics teachers in Croatia, Slovenia, the Netherlands, and Denmark to Lesson Study is the natural continuation of the MERIA project in the sense that we mainly invited teachers to test the scenarios developed in the MERIA project, while it was only towards the end that the teachers were invited to design new materials following the guidelines for inquiry-based teaching modules drawing on the Theory of Didactic Situations (TDS) and Realistic Mathematics Education (RME). According to the evaluations of the project, the teachers appreciated the work with MERIA materials. However, to enable a more permanent impact on mathematics teaching we need to develop further the "infrastructure" for the teachers' continuing design and maintenance of such scenarios, which is part of what Lesson Study offers. **In TIME, as in the MERIA project, the principles behind lesson design will be based on the approaches of TDS and RME**; in Japan, the somewhat similar framework exists, but their historical genesis is quite different, and those frameworks are not so widely disseminated outside of Japan.

Questions for reflection:

- Do you think that the idea of Lesson Study is convincing for the improvement of teaching practice?
- Do you see any constraints for its implementation?

Chapter 2. Basic principles of Lesson Study

The Lesson Study cycle consists of four main phases:

- identification and study of a teaching problem,
- planning a lesson,
- teaching and observing the lesson *in vivo,* and
- systematic shared reflection based on examining observation data with respect to the goals of the lesson.

Stigler and Hiebert (1999), as many other authors particularly in the English-speaking world, consider the process as cyclic, corresponding to a situation where the same Study Lesson is revised and retaught. We have depicted the process in the simple diagram of Figure 3 where the numbers in the diagram correspond to the following activities (which leave open the option of repeating a part of the process more than once):

1) Teaching problem identification and study,
2) Lesson planning,
3) Lesson implementation,
4) Lesson evaluation and review of the results,
5) Reconsideration of the lesson,
6) Implementation of the lesson based on reconsiderations,
7) Evaluation and review,
8) Sharing of the results.

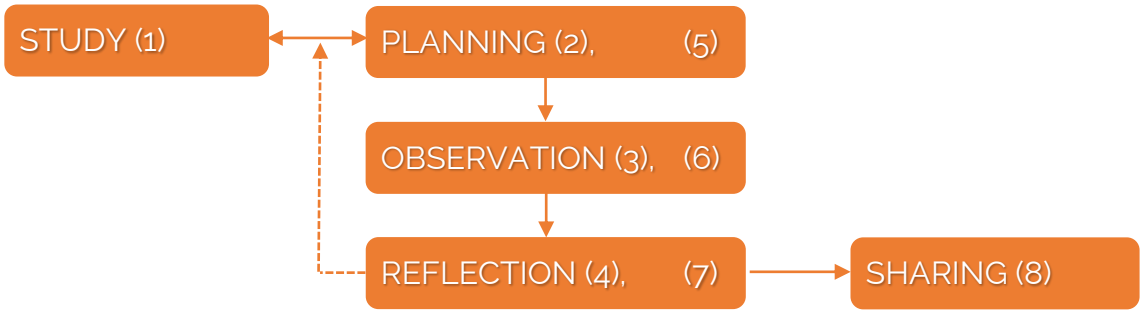


Figure 3. The main phases of Lesson Study, an adaptation of a diagram from Stigler and Hiebert.

In Japan, the sharing can take place in open lessons with an open invitation to all mathematics teachers in the municipality or region of the school, through papers in magazines or books for teachers, and published collections of lesson plans supported by the Japanese Ministry of Education (Isoda, Stephens, Ohara & Miyakawa, 2007, p. 40).

When considering the beginnings of this process, a natural and rich question arises: Where do the teaching problems in mathematics come from? How to identify and formulate an interesting problem? How is it boiled down to a concrete goal for a Study Lesson?

Identifying the teaching problem and learning goals

When identifying the teaching problem, the work of the team of teachers goes beyond agreeing on what topic is difficult to teach to the students or hard for them to understand. One needs to find ***a research theme*** which can drive the Lesson Study process, meaning one or more realisations of the process shown in Figure 3. For example, a research theme could be to improve students' mathematical argumentation and reasoning. This theme arises from the fact that more loose arguments are valid in primary and lower secondary school, where examples can be enough to explain an algebraic relation, whereas in upper secondary schools the teaching aims to develop among students more formal algebraic reasoning, enable them to prove simple theorems and prepare for mathematical reasoning in higher education.

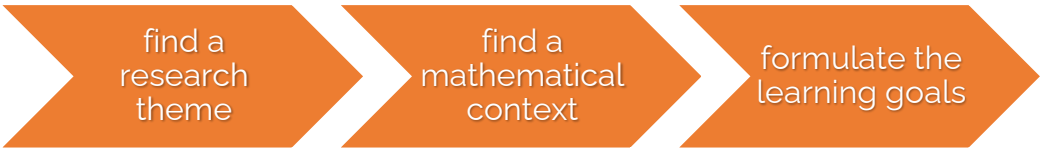


Figure 4. Selection of the teaching problem and identification of the goals.

Thus, to understand the nature of the teaching problem – including students' prerequisites to overcome the challenge – it is relevant to look at how the students have been taught and maybe also previously assessed by national tests or exams. The study is inseparable from formulating good teaching problems. There might be other teachers or researchers who have identified and addressed the teaching problem earlier, thus it is possible to draw on their work as it is shared in interviews, articles in magazines for teachers, reports, research papers, and other media. It is important to strike a balance between obtaining solid grounds for the chosen problematic, and the time available for the entire Lesson Study activity.

Not all kinds of teaching are suitable for Lesson Study. Practising exam exercises might be a common and necessary part of lessons, but it does not as such teach students new mathematics and is not equally relevant to plan and observe in a Lesson Study – there is not much for teachers to study. As part of clarifying and delimiting the teaching problem, it is natural to start looking at ***the mathematical context*** it occurs in. For example:

- Are you considering algebraic reasoning or geometric reasoning?
- Or is it actually a more specific argument related to a mandatory piece of knowledge, such as the formula for calculating the doubling time of an exponential model?
- What is the real challenge you want to overcome?

It is important to agree on this from the beginning.

Once a mathematical context is fixed, the next step is to formulate ***the learning goals*** of the lesson. Formulating the learning goals requires some knowledge about the purpose of teaching, such as what is stated in the curriculum that students should learn: *the target knowledge* (smaller pieces of content knowledge, such as a formula) and *the broader goals* of the lesson (more generic achievements such as competences, skills, possible applications, reasoning). This effort can be supported by other elements of the study, such as revisiting ministerial documents, high stake examinations, treatment of this knowledge in different textbooks, online resources for teaching (including podcasts), and so on. In some cases, it can be relevant to go back to how the topic was addressed when the teacher attended university. Most often, this will not provide a suitable approach for upper secondary education, but it may still help formulate some of the deeper reasons behind the rules taught in upper secondary school and support an in-depth analysis of the teaching proposals found in textbooks. This process may lead to a design of a completely new proposal for teaching the subject, or a thorough revision of existing approaches. In both cases, a deeper preliminary mathematical analysis will again be crucial to design ways for students to realise, construct, and learn this piece of knowledge. In principle, the team of teachers must first have fixed learning goals before pursuing the planning of the lesson; but the preliminary analysis can lead to new possibilities and, hence, modifications of the goal. At any rate, it must be clearly specified what new knowledge the students must have learned when they left the class after the Study Lesson, and the lesson plan should indicate concrete student actions which can help identify whether such learning took place.

An example of an open lesson in Zagreb, Croatia – The beginning

We will consider one example of an open lesson designed and implemented at the beginning of the TIME project, in February 2020. The team of teachers in XV. gimnazija in Zagreb discussed that their students will continue their education in STEM and therefore it is important that during their high school education they comprehensively apply the mathematical language for defining mathematical terms, formulating mathematical statements, and writing proofs. As a problem, they pointed out the students' insufficient precision in communication, as well as reproducing definitions and statements without understanding. They agreed that they want to create a learning environment in which the students will first create an intuitive image of a mathematical concept, then describe the concept in spoken and finally in mathematical language, as precisely as possible.

Within the team, the teachers discussed the mathematical context in which they could create the desired environment. Since the topic of inverse function (exponential and logarithmic functions) is highlighted in the curriculum for the second grade of high school, injectivity was chosen, as it is important for understanding the concept of an inverse function. A discussion of goals followed in which the team formulated the target knowledge: A formal definition of an injective function. The team also defined broader

objectives: Mathematical communication, identification of function injectivity in different representations of functions, application of injectivity.

The selection of the teaching problem and identification of the goals has not been a straightforward process at all. The team has started the discussion with various interesting topics for a lesson. There were a few nice ideas, but some of them have not been focused on the actual problem that the students have. It was crucial for the team to delve on the question: What do we want to achieve and where does the didactical issue lie? Teachers in the team thought about the wider perspective in terms of the students' future and this has led to the formulation of a research theme emphasizing the precise use of mathematical language and the understanding of definitions. From this, broader goals emerged and finally, the definition of injectivity was chosen as a narrower target knowledge because it is usually covered in that part of the year.

Planning and creating the lesson plan

Lesson Study is a joined endeavour, which means that the Study Lesson is a shared experimental device made to help both teachers and students learn – not an occasion for teachers to evaluate and criticize each other. It is a ground-rule when planning the teaching, but it might be important to emphasise this rule to the teachers who have not been brought up in an educational culture like the Japanese system where Lesson Study is an integral part of practising teaching. **A simple agreement like using "we" and "us" instead of "you" and "I", can help new Lesson Study participants overcome the initial unease of opening their classrooms and sharing their lesson preparations with colleagues.** For example, after reading some inspiring idea from a textbook, a teacher may suggest a reformulation of the problem to be given to the students by saying: "I think *we* should change the formulation…" rather than addressing the teacher who happens to be implementing the plan, saying "I think *you* should change the …". This may seem like a naive rule, but for teacher communities where the tradition prescribes that teachers operate more or less autonomously under the constraints and conditions of ministerial guidelines and are never or rarely invited to observe each other, it can be challenging and even initially unpleasant to expose their practice in this way. Therefore, it is relevant from the very beginning of the planning to insist on the fact that the members of the team are in this together – both when it succeeds and when it fails. When observing Japanese or experienced Lesson Study participants, it is possible to experience a more direct tone, simply because the collective responsibility is more evident to them, not because they tend to be rude.

There are different roles for the participants – teachers to take when doing a Lesson Study, which becomes especially important in the observation and the reflection sessions: *Lesson Study Guide, Facilitator, Selected Teacher,* and *External Commentator*. Description of different roles relevant for Lesson Study in the TIME project will be given

along the description of phases in which they become relevant. In Japan, these roles have been developed through the historic development of Lesson Study. The roles are therefore well known to different actors within the school system. For example, they do not have to define what an External Commentator does since all teachers have experienced it in practice. In fact, even pre-service teachers are invited to engage in Lesson Study during their education. We do not have that situation in Europe and therefore we need to be more explicit on the roles and their functioning. Most of the description is based on the authors' participation in Lesson Study in Japan, but also on the international literature related to initiating Lesson Study outside of Japan.

The Lesson Study Guide is the one running and organising the Lesson Study activities in a Lesson Study team. If one school has more than one team of 3-4 teachers, they preferably have one Lesson Study Guide in each team. The guide is planning the meeting activities for the study and preparation of the Study Lesson. An important part of this is enabling all teachers of the team to take part in the Study Lesson. During the study and preparation of the Study Lesson, the Lesson Study Guide leads the discussion and helps the team of teachers to stay focused on the research theme and learning goals of the Study Lesson. It is important that the meetings are well prepared and include contributions from all members.

When planning the lesson with colleagues, it can be tempting to cast aside the teaching proposals, aiming at creating something brand new. This can be very exciting and rewarding work, but it is also relevant to look at existing proposals, analyse these using elements of TDS, and restructure or redesign a problem to provide the students with the potential of reaching the intended learning goals. As TIME is considered a continuation of the MERIA project, it is natural to either draw on the existing modules from MERIA (although they cover very little of the upper secondary curricula), or to draw on the design principles found in the practical guide for inquiry-based mathematics teaching, also from MERIA. So, what is important to decide on when planning the lesson?

In the TIMEplate we list several things to decide on. We already mentioned choosing the *problematic* to be researched and the learning goals, which are split into the mathematical content knowledge (the *target knowledge*) and the competences, skills, or other generic ideas which are deemed relevant to be described as the *broader goals* of the Study Lesson.

For others to take part in the observation and the reflection afterwards, it is important to state the *grade*, the expected *time* frame, and the wider context of the lesson (e.g., a sequence of lessons). This also becomes relevant to share your products and observations with others later in the process. In the case of sharing and observing the lesson plan as a product, it is important to indicate *required materials* – also if it is simply pen and paper, for instance, to point out that students are explicitly expected to solve a problem or complete an argument without the use of technology. If the lesson requires

certain spreadsheets, poster materials, links to web pages, etc., this should be noted, too.

The *problem* the students are supposed to work on must be formulated and reformulated in the planning process until it corresponds to the target knowledge and carries the right learning potential. To secure this, it is relevant to forecast and write down the strategies that students might pursue, based on your knowledge about their prerequisites (what they have learned previously in upper secondary school and even in earlier stages of schooling).

When the problem is fixed, the team of teachers sketches a timeline of the lesson (see Figure 5). This means that they estimate how much time is spent on devolving the problem, individual work, group work, sharing of ideas, testing students' proposals, and teacher's "wrapping up" (the institutionalisation of new knowledge). Thus, the template includes one column for time management, one for teacher actions, and one for student actions. The plan concludes by sketching how the teacher wraps up the lesson, how the main idea should be explained to the class, depending on the strategies promoted by the students. Thus, the institutionalisation is based on and linked to the students' contribution, explaining the mathematics from a higher viewpoint, but still attached to the work of the students. Certainly, this must not turn into a lecture on how the students should have solved or addressed the problem, but the teacher can introduce new terminology etc., which the students can now relate to the work and the results accomplished during the lesson.

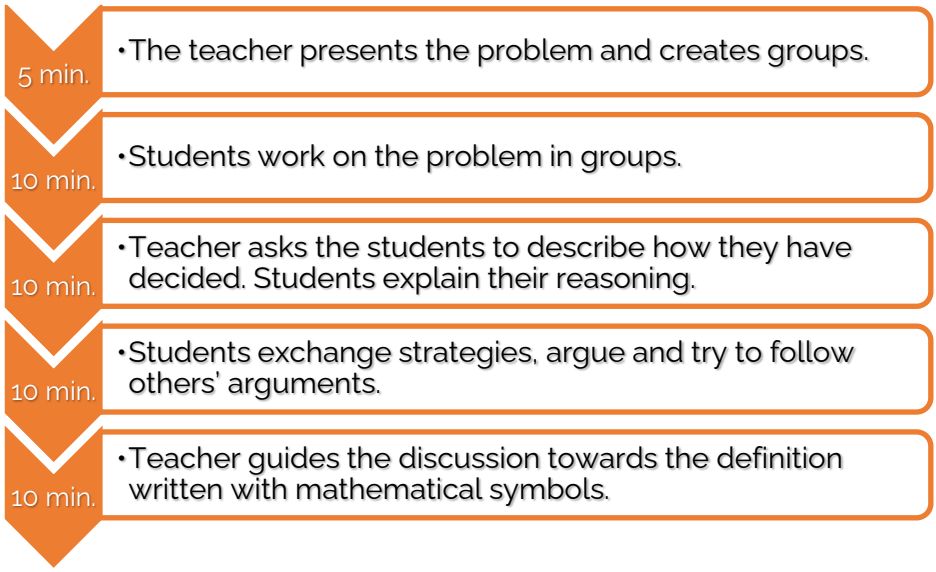


Figure 5. An example of a sketch of the lesson.

While designing the lesson, the team of teachers might face the need to recall or obtain new mathematical knowledge. Or they simply become curious about the details and pieces of mathematics which they have never considered before – probably because

they have never been invited to do so during their education. Then it is relevant to go back and *study* further materials, revisit university textbooks, online lectures, or whatever source that can help pursue this interest or problem. Actually, they might encounter going back and forth between the *study* and *planning* of the lesson (indicated by a double arrow in Figure 3) before settling on every relevant detail for the lesson. This does not necessarily imply many hours of additional work; it is normal that working out the plan and sharing it with others raises deeper questions concerning the content and a need to explore further resources for teaching. Concluding the preparation and drafting of the lesson plan, some elements of the plan (e.g., some student action) crucial for teaching/learning and important to observe *in vivo* may be identified. This can be noted in the plan.

An example from Croatia continues – the design of the plan

The team that has decided to work on a lesson concerning injectivity proceeded to develop a teaching plan. For the students to create an intuitive notion of the injective function, it was decided that they should work in groups and the groups would get different representations of functions: tables, graphs, diagrams, function machine, mapping rules. Finite sets were selected for the domains. The assumption was that different representations would lead to the observation of different aspects of injectivity in the case of finite domains: different elements are mapped into different values, the relation of the number of elements in the domain and the codomain, monotonicity. Students would be given examples of several functions, then they should notice a property that some of the functions have and others do not, organize the functions by the observed property, and explain their decisions. There would be time to compare their reasoning with the reasoning of other groups, discuss the property by which the functions are classified (injectivity), and if necessary, make a new classification in the group. Students would then describe the property of injectivity. In the plenary discussion, they would listen to observations and descriptions of other groups. Based on that discussion and using the students' descriptions, the teacher would write down a formal definition of injectivity. At the end of the lesson, each student would give an example of a function that is injective and an example of a function that is not injective using a representation of their choice. This would show whether the goal of the lesson has been achieved.

The plan was presented to the teachers from the project team and discussed. Changes in the formulation of the task, graphical representations of functions, and the selection of domain elements were proposed. A real-life example was discussed from which students could see the relevance of the discussion of the topic of injectivity, but it was not included in the end due to time constraints and a general focus on the precise use of language instead of mathematical modelling. It was also discussed whether students would agree upon the classification according to the injectivity property and how they would justify that this classification was better than the other possible ones. Some members of the wider

team wondered if the allotted time would be enough for the individual stages. The team studied all the proposed changes and decided which changes to incorporate into the teaching plan. The final plan can be found in the TIMEplate.

The design of the plan was led by the following research questions: "To what extent does the proposed situation with the given representations of functions support the development of the concept of injectivity? To what extent can students formulate the precise definition of injectivity using words or symbols?"

Observation of a Study Lesson

For the observation of the Study Lesson, it is common for the Lesson Study team of teachers to be present. The Lesson Study Guide invites external observers, including an External Commentator and (perhaps more important in a start-up setting!) representatives of the school management, to be part of the Study Lesson and reflection session.

The Facilitator is a member of the team that welcomes the observers (if external visitors are observing), shares the basic rules for observation including whether observers are allowed to take pictures or make a video showing students' actions as resources for the reflection meeting afterwards. The Facilitator may have to stress that observers are invited to move if needed to observe the students closely, for instance, to hear what they say, see their drawings and writing, etc. She/he may have to recall that the observers are never allowed to interact with the students and should not disturb the lesson in any way.

Among the observers, also ***the External Commentator*** is present – he/she should be an experienced teacher with deep knowledge of the subject and ideally with long experience in engaging in Lesson Study. The External Commentator contributes to a wider perspective and an "outside view" at the end of the reflection session.

If the lesson has been tried out earlier and is considered well designed, it can be the main goal to share its details in a so-called "open lesson" with teachers from other schools whether they work with Lesson Study or not. In Japan, this can be seen at events such as Lesson Study festivals and conferences, sometimes with hundreds of participants.

Many researchers have studied the mechanisms of Lesson Study and what this inquiry into teaching practice reveals and affects. One important point for observers is that "… rather than observing teaching, they observe pupils learning in the context of being taught. This process of jointly observing learning in the context of teaching and learning in a lesson is the essence of Lesson Study." (Dudley, 2015, p. 10). For many teachers, it

can be interesting to observe the practice of colleagues, including their various strategies to interact with students. Still, the focus of Lesson Study should most certainly be on the interrelation between teaching and students' learning in the context of a specific lesson. This again might seem obvious and naive, but to observe the dynamics between teaching and learning in real-time is different from commenting on a colleague's teaching skills or the nature of student actions independently. Therefore, all observers must have the lesson plan – usually a short version of about 1-2 pages – early enough to know what to observe and what to expect from the students.

A ground rule for observation is not to interfere with the teaching.

Initially, the observers stand along the walls of the classroom (behind or on the side) in order not to disturb the teacher and the students. They will stay there, take notes, take pictures (if this is agreed upon before the lesson, also concerning GDPR rules), observe the devolution of the problem, etc. When the teacher invites the students to act individually or in groups, it is common for the observers to move more freely in the classroom and observe students' work (discussing, sketching, writing, etc.). Experience shows that it can be difficult for observers not to interact with the students the first time they experience lesson observation. But it is a non-negotiable principle that they should not. If the students are new to this as well, the Lesson Study Guide may announce, before the lesson begins, that the observing colleagues are *only* there to observe, not to teach or otherwise interact with students who should simply ignore their presence and "behave as usual". It turns out that they do so much more easily than most first time Lesson Study observers would believe; the hard part is for the teachers – observers to concentrate fully on observation.

The Selected Teacher is a teacher from the Lesson Study team teaching the class. Most often it will be her/his class. Experience shows that, when teachers get used to Lesson Study, they like to take on this role, since it implies that they are getting peer-feedback on their realisation of the prepared plan. In TIME it is important that every member of the team of teachers doing Lesson Study gets experience with both, the role of observing and the role of teaching the Study Lesson.

The Selected Teacher follows the lesson plan as closely as possible, but if unforeseen events occur, such as students not understanding or engaging in the problem, the time estimates prove wrong, or the technology planned for the teaching will not work, then, of course, the Selected Teacher should adjust the lesson accordingly. In the reflection session, the Selected Teacher is the first to share her/his reflections, including explaining if and why deviations from the prepared plan occurred during the Study

Lesson and the reasons for changes to the plan. It may not ruin all potential for learning from the experience; in fact, it may lead to necessary adjustments to the plan, particularly in the first trials of it.

Other teachers from the team are involved in all phases of the Lesson Study, including study and planning. During the Study Lesson, these teachers play a special role knowing the plan in more detail and they can bring an insider's perspective to the observation and the reflection sessions.

The main focus of the observers is on the students' learning in relation to the goals of the lesson.

Observers should try to follow the chain of thoughts of at least some students to gather relevant data to share at the following reflection session. Different techniques for observation are developed by teachers as they become more experienced observers, but it is important to note:

- how the devolution is received by the students,
- what actions they take,
- what hypotheses they formulate,
- how the validation engages different students.

It is worth noting even small differences in students' actions and formulations. Do these indicate a different understanding of the problem, different prerequisites, or different learning? What is the role of students sharing hypotheses, strategies, solutions, or whatever is relevant in the specific lesson? Does it promote or reduce students' learning? Are there unexplored potentials of slightly overlooked student actions?

It is important not to simply criticize the teacher or the teaching but to focus on the relation between the lesson plan and observations from the classroom, which can indicate the extent to which the explicit goals have been reached.

As a suggestion promoted by Dudley (2011) for the implementation of Lesson Study in the UK, one can attempt to describe some *prototype students*, their prerequisites, and corresponding anticipated strategies for solving the problem. These are described in the lesson plan, for the observers to identify such students and describe how they engage with the problem and the milieu. This is a way to focus on planning and observation of different types of students and their learning. The idea of describing, identifying, and following the prototype students, links to a more Western idea of differentiating the teaching according to the different types of students in the classroom and it is not used in Japan.

An example from Croatia continues – observing the open lesson

The open lesson in XV. gimnazija was observed by about 15 Croatian teachers and 15 TIME project team members from Denmark, the Netherlands, Slovenia, and Croatia. The lesson plan was introduced to the observers from the project team the day before the lesson and they had a chance to comment on the design.

Immediately before the lesson all observers have been welcomed by the Facilitator and reminded not to interfere. The lesson was held in English and it started with a short introduction by the Selected Teacher. Papers with different functions were handed out to students. Each group of students had the same set of functions but given in different representations and printed on different coloured papers. Once the students started working on the assignment, observers were allowed to circulate the classroom, come closer, and listen to students' discussions.

Once the students made their first classifications, the observers moved away back to the walls of the room and the students hung the papers on two ropes – one rope contained the functions with the property and the other was designated for the functions without the property (see Figure 6). The Selected Teacher asked one member of each group to state the property which they used to classify functions into two groups.

Figure 6. The papers hanging on the rope.

After a short discussion, the students were asked to write in words the precise formulation of the property, which the teacher named injectivity. The observers were once again allowed to circulate and observe students' work. The students presented their formulations orally, and one student wrote all formulations on the blackboard:

- Injective functions give different images to different elements.
- Any number will have a different image.
- Every value in the codomain must have only one element from the domain.

The teacher-guided the discussion and tried to engage all students. Those students who did not speak before were now called out to explain the property in their own words. During the discussion on various formulations, the teacher addressed the imprecisions and misconceptions in some formulations. In the last minutes of the lesson, each student had to write on a piece of paper one example of an injective function and one example of a function which is not injective. The observers had the chance to read the examples once the students have left the room.

This shows various ways in which a lesson may be adapted for observation: observers approach students during group work, students present their work on the blackboard (or a piece of rope, posters, etc.), students' papers are left for analysis, many students are asked to speak, and so on. These are all didactical aspects that illustrate the flow of the lesson and what kind of information has been made accessible to the observers, but many details of the lesson, in particular those related to the content, could not be presented here. We will mention some of them in the comments on the reflection session, but many more could be found in the TIMEplate, section Practice report.

Reflection session

The reflection session should, if possible, be held immediately after the lesson is taught, preferably in the same classroom in which the Study Lesson took place, with a short break allowing the students to leave the classroom. In that way, the used blackboard and student work are readily available. The Facilitator is the "host" of the reflection session with the agenda as in Figure 7. If necessary, it could be the Lesson Study Guide taking this role.

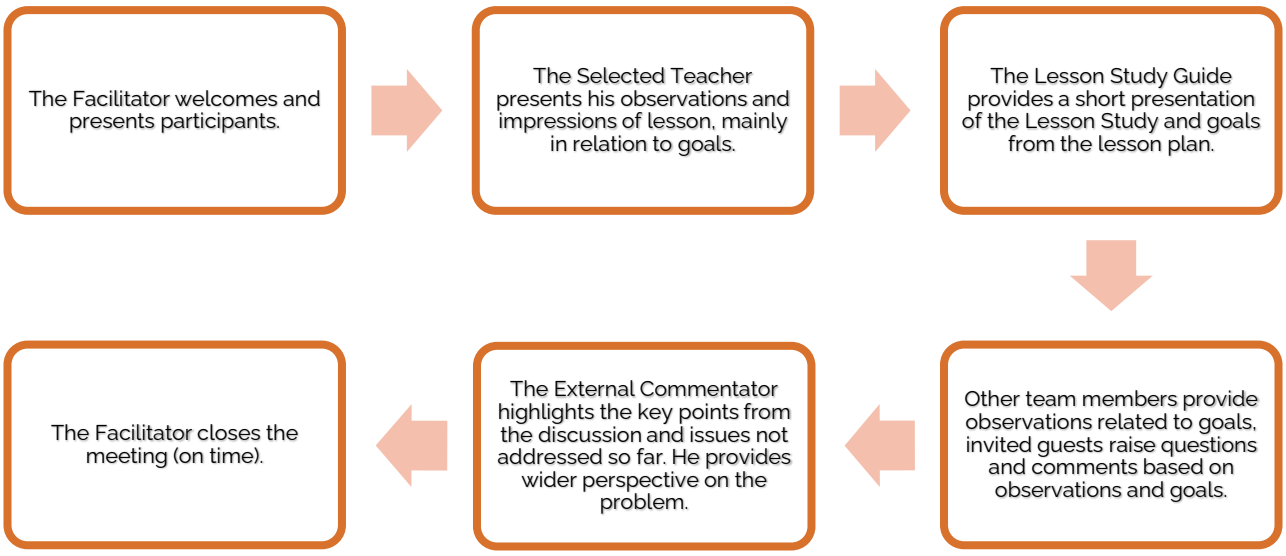
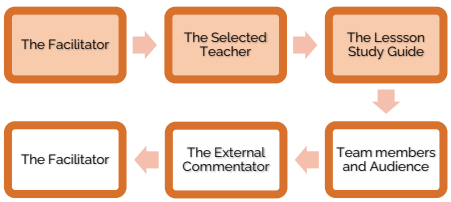
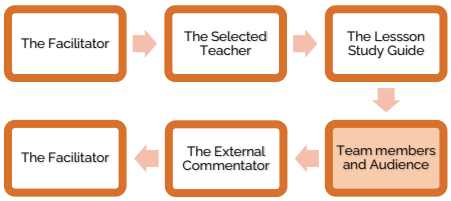


Figure 7. Agenda for the reflection session.



After a short welcome (and the presentation of participants if the session is not purely internal), the Selected Teacher presents the impressions from her/his special point of view. She/he also provides some explanations if she/he had to adjust the teaching compared to the initial plan. The Lesson Study Guide

provides a short presentation of the Lesson Study, the lesson plan, some rationales, and specific issues the team of teachers would like to discuss with the observers.



After this, the other teachers from the team provide their inputs based on their notes from the observation, pictures, videos, etc. If other observers are present, they are given the opportunity to provide their observations, contributing an "outsiders' view" on the lesson, concrete data on the students' learning as a result of the teaching, the problem, and the teachers' questioning. It is common for outside observers to ask questions as well, which may then be answered by the Selected Teacher, the Lesson Study Guide, or another member of the team.

Throughout the reflection session, the Facilitator assures that the discussion stays on track and ends on time. She/he may structure it by inviting comments and questions on specific issues from the lesson plan. During the first experience with Lesson Study, it is particularly important that the Facilitator invites comments or questions that focus on observations that relate specifically to the teaching problem and the goals of the lesson plan and that the comments are based on specific lesson observations; comments should not merely be assessments of the students nor the teacher but should stay focused on the goals of the Study Lesson. This is not the easiest task, but everyone learns more from the session when these points are observed.

Figure 8. On the panel, in front of the rest of the observers, sit the External Commentator, the Selected Teacher, the Lesson Study Guide, and the Facilitator.

The External Commentator is invited to participate in the process by the Lesson Study team based on their confidence and trust that this person can contribute to their study. In internal meetings, the External Commentator may be the only one to provide an "outside" perspective. This person carefully studies the lesson plan beforehand, participates in observation, and is present at the reflection session but only speaks at the end. Before the session, it is also necessary for him/her to think about potential comments based on the experience and knowledge of the problem the lesson plan indicates. During the session, she/he concentrates on noting important observations made by other participants and finally decides on what to stress in his final comments. The rationale for including this person is to share a wider perspective and knowledge

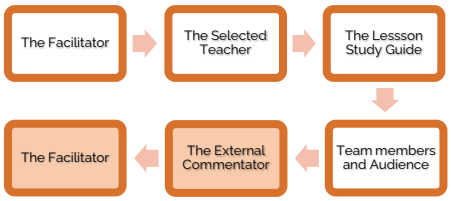
on the educational system, Lesson Study, and mathematics education. This means that the External Commentator ideally has a broad knowledge of mathematics to be taught in the lesson and on how to design inspiring inquiry-based teaching.

In Japan, External Commentators are typically experienced Lesson Study Guides with hundreds of Study Lessons on the resume. They can also be university researchers (we would say, a didactician of mathematics), ministerial experts engaged in developing upper secondary mathematics education, textbook authors, or others having more in-depth knowledge about upper secondary education. The most popular External Commentators in Japan are invited to contribute to Lesson Studies all over the country, sometimes several times every week, and contribute to advance a widely shared professional practice and knowledge in the country (Takahashi, 2014). Of course, when we develop Lesson Study in another setting, this expertise has to be built up as well, but experience shows it is important that the reflection session ends with at least a summary and some useful perspectives given by a person who is entrusted with this specific role and sticks to it. The final comments, which may last just a few minutes, based on carefully following – but not participating in – the observation and reflection sessions, are necessary to put together the collective learning of all participants.

As indicated earlier, there are different "sizes" of Lesson Study varying from a small group of teachers in a school forming a Lesson Study team, even a single teacher in some cases. But in a school, there can be several teams of teachers working in parallel and at special occasions join each other's Study Lessons. In "open lessons" it is also possible to invite other teachers from the school or district who are interested in learning about the work carried out in a particular team of teachers. Open lessons in Japan are also common at the regional or national level, with hundreds of observing teachers, e.g., in a gym hall (in this case, observation "between the desks" of students is evidently not allowed). Regardless of the size and format, it is crucial that the visiting teachers respect the preparation and work of the team of teachers. It is an honour to be invited, so respect the hosts, and enjoy it!

In Japan, but also in other countries, Lesson Study is a format used to connect pure mathematics courses, mathematics education courses, and the mathematics teaching practice. Pre-service teachers are often invited to take part in open Study Lessons. It can be an advantage to let such participants reflect in groups before sharing their ideas in plenum with other participants. The pre-service teachers might have interesting new or different approaches to the teaching due to the courses on mathematics education they follow. While they have little experience, one should embrace the resource that they represent, to ask different questions and provide different comments from those given by experienced teachers.

During the whole session, it is important that the Facilitator keeps the discussion on track, which is often done with a loose hand especially in the setting of experienced participants; also, the time frame is not to be exceeded.



As a wrapping up of the reflection session, the invited External Commentator highlights the key points addressed in the discussion and points out relevant issues not addressed so far. This person also provides a more general perspective on the problem addressed by the team of teachers, to share a deeper understanding and sometimes research-based knowledge related to the problem, leading to suggestions for further study and how the participants can address this problem in the future.

An example from Croatia continues – the reflection session

In the case of the lesson in XV. gimnazija, the reflection session was organized in the same room as the lesson. Once the students have left, the Facilitator has opened the session and the Selected Teacher presented her experience. The lesson went according to the plan and the students confronted expected issues, e.g., the notion of surjectivity has popped-up and the teacher had prepared an example of an injective, non-surjective function. The teacher noticed that one group did significantly better than the others and revealed to the public that she has guided the discussion accordingly. Domains of the given functions were finite sets, and students were not expected to show a misconception by considering that functions should always have the set of real numbers as their domain. It was decided that this would be addressed in subsequent classes. The teacher expressed that she was satisfied because it seemed that the lesson required optimal effort from the students to reach the target knowledge.

The Lesson Study Guide spoke next and recalled the research questions that the team of teachers posed for this Study Lesson. The lesson had two parts and in the first part, we could see that students have organized the papers as expected, so the concept of injectivity was recognized. In the second part, the students worked on the precise use of mathematical language and have been successful, but certain details had to be discussed and corrected. The examples that the students wrote at the end of the lesson showed that it might be very interesting to consider which representations certain students used. Some of them used the representation in which the problem was initially posed in their group, while others switched to a preferred representation.

There were 18 questions from the public and these were answered by the Selected Teacher. The questions have touched many different aspects of the lesson and decisions made by

the teacher. One may try to notice which questions relate to the research questions posed by the research team:

- *What was the students' pre-knowledge? Did they already use the representation with mapping diagrams?*
- *Did all the students realize that they all have the same functions?*
- *Did you feel the pressure of time or that you will not come to the end of the planned lesson? Which choices did you make because of time?*
- *You have prepared an additional example of a non-surjective function. Why did you not give it immediately?*
- *Do you think that students looked at the functions which others were hanging on the rope or think only of their own?*
- *What would you do if none of the groups discovered the property? Would you leave out the representations that did not lead to the property next time?*
- *Why did you dismiss the definition "each element of the domain has only one image"?*
- *What would you expect if students were given different representations?*
- *In the mapping diagram representation, in neither example did the lines intersect. Do you think examples with crossings would be successful, too?*
- *I like the ropes; we could all see the proposed classifications. Do you think the blackboard should have been divided into more parts so that the groups could write in parallel?*
- *Why didn't the students go to the board, but you chose one student to write what the others were saying?*
- *Students had to define something that they were not motivated for. How do they feel in the end? Are they aware of the validation rules? Maybe you could have asked for an example.*
- *I observed that the students struggled to look at functions as objects. Was it the first time to consider them in this way?*
- *When you asked the students to write examples of injective functions, did they use the representation in which they understood the concept?*
- *How does a reaction to a mistake affect learning? Should you have addressed the misplaced paper earlier? Could students follow?*
- *Could you comment on one student saying, "There is no right answer"?*
- *The group that I have observed commented that the teacher said not to look at the type of function. They have discovered that there are positive and negative values but felt there should be more. How do you comment?*
- *There was a confusion about the domain and the mapping rule. Some students said that $f(x) = x^2$ was not injective, but the role of the domain was not settled for everyone. Would you address this issue in the lesson?*

The discussion lasted for about one hour and has been concluded by the comment from the External Commentator. It was three-fold. From the point of view of generic goals (different modes of communication), it was valuable to see students using different representations, both in informal and formal language and to express themselves orally

and in written form. From the didactic point of view, we have seen that the question posed by the teacher did not evoke the desired answer. Furthermore, the milieu could not provide validation of the answer and the teacher was the authority declaring what is right. Nonetheless, it was a situation showing a typical day in (offline) school with a rich mixture of verbal and non-verbal (even implicit) aspects. From the mathematical point of view, the lesson dealt with the topic that is hard even at the level of university students and has been an inspiration of research in mathematics education around the world, especially in the context of calculus courses and developing students' ability to conjecture and produce proofs.

More details about the reflection session could be found in the TIMEplate, section Practice report.

Revision of the lesson

In some cases, the lesson plan is revised (5) and implemented again (6), which again will be followed by another reflection session (7), see Figure 3. When revising the lesson plan, it is relevant to consider the suggestions of the External Commentator and the comments of other observers. This means, to revise the plan using their comments to further the learning of the students in the next class. In principle, the process can be repeated several times. But it is important to try to distinguish between class-specific observations and more fundamental issues in the lesson plan. To study the chosen problem, the solution is not simply to discard any hypothesis or goal that was not realised in a given trial or to reduce the time of action for students. Instead, one should figure out the challenges and improve the design, whether the same teacher teaches a new class, or another teacher from the team does it. The important thing is that the team of teachers takes the opportunity to revise the lesson and observe the effects on students' learning. Of course, different students and new situations lead to different outcomes – this is not comparable to controlled experiments in a laboratory. Indeed, qualitative investigations of students' learning of mathematics in a social setting is a much more complex subject – but still, the reality which mathematics teacher's knowledge is all about.

Lesson Study cycle ends with writing a Practice report. Sharing the experience is one of the crucial aspects of Lesson Study, to make the results of the experiment known to others, but also to make them explicit for oneself and thus improve their practice. We return to this from a practical perspective in the next chapter.

Chapter 3. Practical issues for Lesson Study within TIME

In this chapter, we comment on organizational issues related to setting up Lesson Study in the context of Project TIME. This means we discuss establishing small communities of teachers in some European countries, with a focus on secondary school mathematics and implementation of inquiry-based mathematics education.

Project TIME: Using Lesson Study in new contexts

We all know that there are very different kinds of teaching in mathematics classrooms of upper secondary schools, and not all are equally well suited for Lesson Study. When students are preparing for the written exam by practising old tasks and exercises, there is not a lot of opportunities to devolve a problem nor a milieu to explore and inquire about. The same can be said of lecture-style teaching, where the teacher goes through an important but complicated proof, or when homework is returned and discussed.

What makes more sense for Lesson Study is to engage inquiry-based mathematics teaching (IBMT). The main goal of project MERIA was to promote IBMT and support teachers in implementing it by providing teaching scenarios, practical ideas, and professional development activities. All of this was strongly initiated and put forward by the project team. Hence, the project TIME as a follow-up project has a goal to investigate to which extent could such resources and changes be created by small communities of teachers inspired by Lesson Study. In TIME, we suggest that participants initially draw on the materials provided from the MERIA project, with practical designs crafted from the perspectives of the Theory of Didactical Situations (TDS) and Realistic Mathematics Education (RME). Such theoretical framework may contribute to the theorization of Lesson Study. For example, **TDS has the potential as both a design tool and an analytical tool where the perspective of the teacher offers "a detailed analysis of how both students and teachers develop their knowledge in various phases of Lesson Study"** (Winsløw, Bahn and Rasmussen, p. 139). Nonetheless, other inquiry approaches can be suitable, too.

Mathematics Education -
Relevant, Interesting and Applicable

Teachers' Inquiry in
Mathematics Education

Figure 9. Project TIME is a follow-up to project MERIA, both promoting inquiry based mathematics teaching grounded in RME and TDS

In Japan, Lesson Study is often representing teaching designed from the perspective of an open approach. This means that students are presented with a problem being genuinely open to several strategies for its solution. This furthers the mathematical communication and reasoning in the classroom in the sense that strategies and answers

naturally need to be discussed and compared to validate, not only that the answers and the strategies are correct, but also to compare them to see if some are smarter, more direct, or general. While institutionalisation (wrapping up the lesson by relating to all the students' contributions and at the same time present the official knowledge, the institutional answer to the problem) is crucial in TDS, open approach lesson may end more openly, merely summarising different strategies developed.

Most of the experience with Lesson Study is in primary and to some extent lower secondary school. The experience in upper secondary schools, which is the target group for the TIME project, is more limited. Some differences may provide challenges and opportunities.

Upper secondary schools are inhabited by adolescents. The students might be less open to a sudden change to the normal routine of lessons. Older students are more aware of themselves and feel less comfortable about being observed. On the other hand, these students are better at reflecting on a lesson and their contribution to it, and the change in the routine might be an opportunity to employ this in the classroom. This may contribute both to the lesson itself and the reflection on the lesson.

The teaching and learning in upper secondary schools are in most countries heavily concerned with important final exams. Students may expect traditional exam preparation in classes. Changing a lesson to inquiry-based teaching could be considered by the students as irrelevant since traditionally exams do not require much inquiry. This also forms a challenge to use Lesson Study to design lessons that prepare for the exams using inquiry. Generally, teachers have the responsibility to explain to students how the Study Lesson contributes to the preparation for tests or exams. In that way, the students are engaged without a feeling that they are "losing time".

Secondary schools are sometimes larger than primary schools and the curriculum is divided into more subjects. This provides more opportunities to work together in various teams. Teachers are specialized in their subject, but collaboration across subjects can be fruitfully explored. Moreover, the content is much more complex in secondary schools and students might show bigger differences in the style of learning and level of achievement.

The project team intends to investigate all these and more specificities in the implementation of Lesson Study in secondary schools. Hence, these topics will also be addressed in more detail in the reports.

Infrastructure and initiating Lesson Study

In Japan, Lesson Study is an integral part of the school system. In a school, most teachers are used to participate regularly in Lesson Study teams, in particular in elementary schools. This means those schools will have experienced Lesson Study Guides and timetables are adapted to allow teachers to collaborate in several Lesson Studies per year.

Outside of Japan, Lesson Study is still in its early stage and the way it is implemented in different cultural contexts around the world differs. We now find a lot of simple models of the Lesson Study process and step-by-step descriptions that miss some important aspects of doing Lesson Study, e.g., teachers' and students' learning in the different phases of the Lesson Study process. **"For Japanese educators, Lesson Study is like the air they breathe"** (Fujii, 2018), but for a novice of Lesson Study, you have to learn how to breathe and often also why you breathe before it becomes natural to inhale and exhale. Winsløw, Bahn and Rasmussen (2017) stress that Lesson Study requires more precise models of Lesson Study based on theoretical frameworks. The point is not to construct a complete description, but to peel off the outer layers of Lesson Study as well-understood methods and instead come to understand the cultural elements of Lesson Study. Clivaz and Takahashi (2018) find that the lack of a comprehensive understanding of the core in Japanese Lesson Study is the main cause why it is difficult to establish sustainable Lesson Study outside Japan, and Miyakawa and Winsløw (2017) argue that Lesson Study is but one element of a comprehensive Japanese "infrastructure" for the professional development of teachers and Lesson Study cannot be understood when only studied in isolation. Instead, we must study what Miyakawa and Winsløw call *paradidactic infrastructure* in which the teacher works when he is not teaching, e.g., the interplay of textbooks, curriculum documents, participation in professional development courses, and teacher conferences.

In Europe, Lesson Study is still rather unknown as a practice and it is mostly initiated from universities. This means that, whenever you want to organize a Lesson Study within a school in a European country, you will start from scratch. The success of initiating Lesson Study in a country and/or a school depends on many factors, such as:

- How are schools organised and managed (legislation, governance, …)?
- How is the teachers' work organised? What is required of them? What is rewarded?
- How are the teachers educated (pre-service, in-service)?
- What traditions exist already for teachers to exchange lesson plans and similar resources?

A school that is new to the concept of Lesson Study will have to decide on the scale on which they want to experiment. An experimental period of a year or more, with more than one team, will have a bigger impact and will be a better approximation of the full

potential of a Japanese experience than a single Lesson Study cycle with a single team. Nevertheless, the latter is also usually seen as rewarding for participants.

As the team of teachers forms, it is crucial to have the support from the school principal and to start with long term planning of the Lesson Study cycle. In practice, first matters to consider are the size and the members of the team, time available for study and meetings, and the schedule for the school year in which the Lesson Study will take place.

Lesson Study may be primarily viewed as a form of teachers' professional development, with a central feature that one works together in teams of teachers. It is all about collaboration and learning from each other. Listening to and learning from colleagues is the main point of Lesson Study – one does not contribute or learn by imposing their ideas on everyone.

For Lesson Study novices, working several cycles of Lesson Study in the same team has reportedly been advantageous, e.g., to generate trust and solid routines. Each team member should feel the importance of the goals and ownership and responsibility concerning the chosen approach and materials.

In the contexts where Lesson Study is new, to succeed in its implementation it is important to involve school management and have its support. Involved management will help to implement and support future cycles of Lesson Studies. It is important to foster a positive atmosphere, where teachers are treated as equals and ideas are welcomed from everyone. Teachers need to be dedicated and willing to discuss each other's ideas and contributions. Positive criticism should also be encouraged. Sometimes, choices and preferences in teaching can be a personal matter, so a gentle, relaxed, open-minded attitude is essential for a successful Lesson Study experience. In Japan, the tradition of so-called *Bonenkai* helps to foster good social relationships between Lesson Study participants. This involves joint drinks and snacks after a Study Lesson. This crucial aspect should not be forgotten to be implemented along with the other elements of Lesson Study.

Questions for reflection:

- Do you discuss your lessons with colleagues? What do you usually discuss?
- Which characteristics of collaboration do you find the most important?
- What are the conditions like for the implementation of Lesson Study in your country or school? Which issues and opportunities do you see?

How to set valuable goals and make an effective plan?

Lesson Study is a goal-oriented activity. A group of teachers sets out on a joint journey that resembles to research with certain questions, hypotheses, and methods that they agree to explore. This is fuelled by a common desire to improve their practice and achieve a certain goal.

So, **what makes a valuable goal for a Lesson Study? For a wider educational community, it is interesting to hear about experiments and practices that lead to improvements of the teaching and learning process.** Hence, it is desirable to formulate goals that relate to the teachers' experience of current challenges; goals that meet shared concerns of the profession (e.g., a new curriculum); and, finally, goals that can be meaningfully realised (at least in part) during a lesson.

It can be a real challenge to formulate shared, relevant, achievable goals, in particular for novices. One issue is that schedule and time constraints can interfere with goals, but the team should keep in mind that goals can be refined during the study and planning phase. It is the Lesson Study Guide who has an important role in insisting on this point and supporting it. The first meeting is a good opportunity to set up goals and planning that support each other. Different members of the team will bring different perspectives on the table, but they should agree to work with a common goal that is motivated by their practice and, in particular, by the issues that their students are confronted with.

In Japanese schools it is common to have a research theme (as a wider goal) for Lesson Studies: a grand goal pursued for several years, often shared across disciplines and Lesson Study teams within a school. Examples (taken from "Leading Lesson Study" by Stepanek et al.) are in Figure 10.

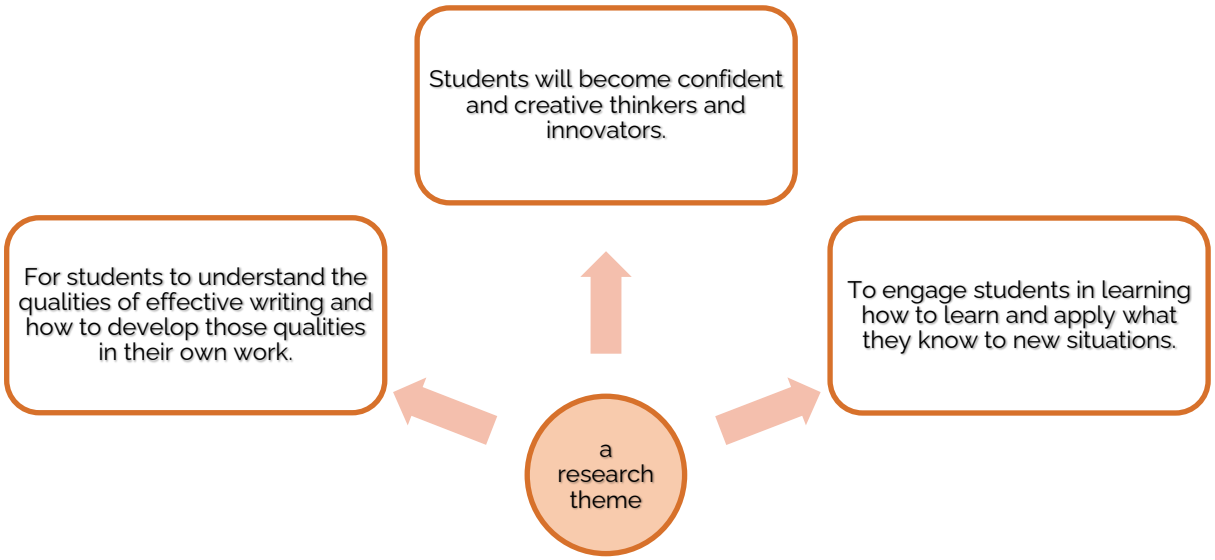


Figure 10. Examples of a research theme (taken from *Leading Lesson Study* by Stepanek et al.).

Obviously, such goals are based on a broad consensus of current challenges and learning needs. **For the TIME-project the aim is that all teams at one school decide on and pursue the same research theme for the whole duration of the project!** It could be mainly/only pertaining to mathematics, but maybe choosing a more generic goal, like examples above, could spur interest in Lesson Study among other colleagues.

Questions for reflection:
- Does your school have a mission statement?
- If you are already a part of a team of mathematics teachers, have you discussed common values and issues?

Lesson planning requires both generic and specific goals. Generic goals refer to skills that are transferable to other domains, while specific goals are more concrete and related to a certain piece of mathematical knowledge. In that sense, we also talk about broader goals and the narrower target knowledge of the lesson. For example, from more generic to more specific, one could formulate goals like in Figure 11.

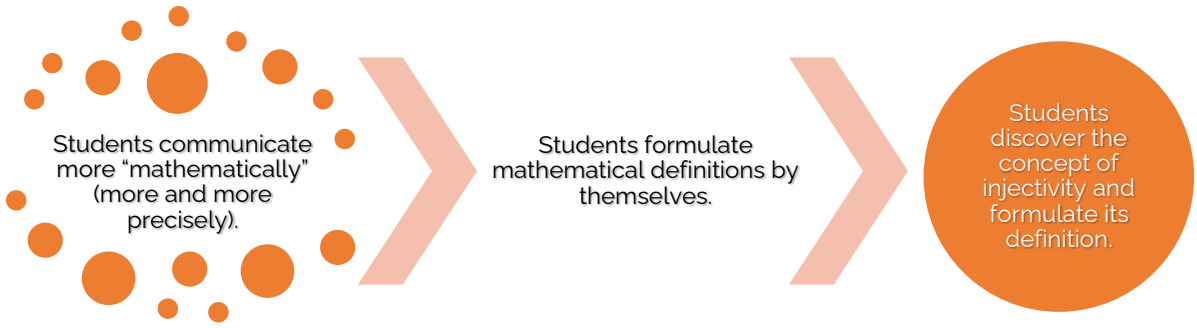


Figure 11. An example of different goals. From a more generic to a more specific goal.

Once generic and lesson specific goals have been set, the team begins studying relevant material to support the planning: syllabus, textbooks, online resources, etc. – perhaps most often individually, but it could be organised as a collective study period for the whole team, at school or outside. A (second) meeting of the team must be held to exchange ideas about current and innovative approaches to the subject of the lesson. Although planning is done differently in different teams, often main ideas, such as the choice of a teaching approach and/or problems for students, are decided by the team together. Before or between planned meetings, one team member (e.g., the Lesson Study Guide or the Selected Teacher) should prepare drafts for a detailed lesson plan which is then thoroughly discussed and fine-tuned by the team as a whole. It is usually not efficient to do extensive writing in a team, so it is suggested to decide who will write and who will comment on the drafts.

It can be a challenge to support the study phase of the Lesson Study, in particular in countries where Lesson Study is not embedded in the culture. **In Japan, outcomes are properly documented in accessible ways for teachers to consult, e.g., in dedicated journals.** In other countries, teachers are dependent on their local (mathematics) teachers' magazines, or the (less accessible) academic educational research literature. For this reason, a team might include researchers in mathematics education working outside of the school. The Lesson Study Guide must insist on the value of both literature study and written plans – without these elements, there is no Lesson Study.

Sharing the best ideas

Mathematics teachers' knowledge is, as other professional knowledge, difficult to share completely and effectively through written text. We have already mentioned the Japanese practice of open lessons in the district, in the region, or nationally; it aims at sharing and building professional knowledge in direct relation to a common observation of practice. Miyakawa and Winsløw (2013) analyse a case of open lessons and "lesson festivals" (in a primary school), but it is otherwise little documented in the English language literature. In Japan, teachers also share their good ideas in magazines published to share reports from Lesson Study and similar activities (cf. Miyakawa and Winsløw, 2018). Professional journals for mathematics teachers at different levels do exist in most Western countries and may perhaps be used to serve a similar purpose, in particular, to share materials from TIME.

In Japan, teachers also share findings from Lesson Study in 2-6 pages reports, called "Practice report": *Lesson Study teams produce reports about their research lessons. For school-based groups, copies of the reports are kept at the school and are available for other teachers to use. Some reports are also published and sold in bookstores (…). The report serves as a means for teachers to reflect on their learning from lesson study, as well as to capture and share their knowledge.* (Stepanek et al., 2007, p. 133).

A Practice report should present

- the teaching problem and/or goal of the Lesson Study,
- a brief description of the study to formulate the problem the students worked with,
- materials including pictures of its functioning,
- main student strategies and how and if they were realised in the classroom.

Finally, the teachers provide some of the reflections and discussion from the Study Lesson and the reflection session.

We suggest that participants in TIME share their work and report on it by using the guidelines provided in the TIMEplate. For the possible sections see Figure 12. The final report on a Lesson Study includes some elements of the lesson plan as well as some of the observations and reflections resulting from the actual research lesson. Pictures of key artefacts from the lesson, including students' work and blackboard writing, can help the readers who were not present grasp some of the key points.

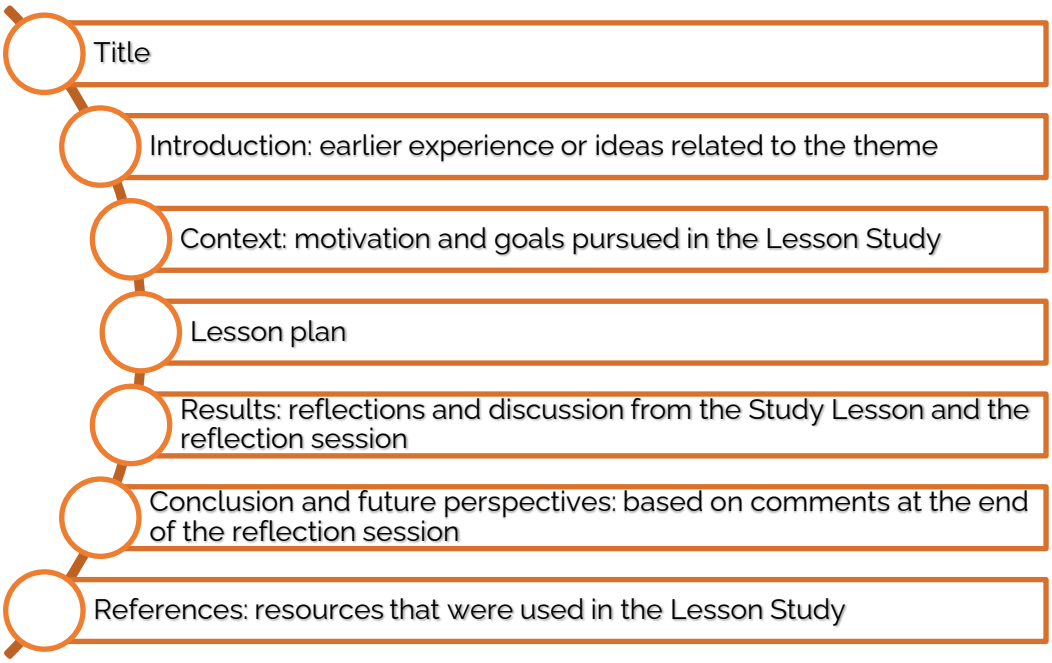


Figure 12. Possible sections of the Practice report.

Clearly, there is some difference in what one will emphasise in a report mainly done for an easier recollection of the lesson for the team involved, and in a report destined to be read by many other teachers (like, to be published in a journal for teachers).

However, it is a good rule of thumb to make the report as accessible to "outsiders" as possible, both because this makes our reasoning and reflections clearer and more explicit, and because the document becomes more useful – even to the team members themselves, as we tend to forget crucial details after some time has passed.

Thinking of publishing or sharing more widely the results of the Lesson Study is meaningful not only to allow colleagues to benefit from the ideas and observations developed but also to help others get to know about what Lesson Study is and what can it achieve. Well edited Practice reports – sporting striking ideas and observations - may, in the long run, be more important entry points for teachers to the practice of Lesson Study, than dry and abstract treatises like the one which ends here.

Appendix: A Lesson Study in the Netherlands

To illustrate some of the challenges of implementing Lesson Study in Europe, where paradidactic infrastructure is still limited, we present an example that took place in the Netherlands in 2017 – in addition to the Croatian example from Chapter 2. We describe the events until before the Study Lesson.

In the Netherlands, there is a paradidactic infrastructure for Lesson Study called LessonStudyNL, which consists of a consortium of educational researchers and teacher educators from universities across the country. There is growing interest in Lesson Study in the Netherlands in the last 10-12 years, with more individual researchers investigating its possibilities, in some cases travelling to Japan to learn about it. By now the Netherlands has some national experts on the subject like Nellie Verhoef, Siebrich de Vries, and Sui Lin Goei. LessonStudyNL was founded in 2016 and has the ambition to implement Lesson Study more widely and sustainably across Dutch education. The Lesson Study below was developed as a public lesson for the first national LessonStudyNL Congress.

The initial organisational features of this Lesson Study have some characteristics that will be common for many countries outside Japan beginning to work with Lesson Study. An enthusiastic teacher was found, but – even though there was some infrastructure on the national level – there was no infrastructure for this at the school level. Other available and interested mathematics teachers had to be found; a meeting with school board members had to be organized to explain what Lesson Study is and ask permission. The LessonStudyNL members explained the benefits and the necessary time investment. The board members were very positive about the opportunity for teachers within the mathematics department to collaborate on the development of teaching ideas and practice. A team had to be completed, which was not straight forward: as in every school, some teachers are easily enthusiastic about novelties, there are so-called mid-adopters, and there are those who close the ranks. Fortunately, a mid-adopting teacher was found interested as well; one researcher joined the team as a regular member and one researcher was the Lesson Study Guide.

A first Lesson Study meeting was initiated by the Lesson Study Guide. In this meeting, dates were set for the developmental meetings and the Study Lessons. The teachers decided who was going to be the Selected Teacher. Then they decided what would be their main themes. The first theme was the relations between graphs and equations. The relation geometry-algebra is a major theme, insight and method in mathematics, and present in the curriculum. The teachers found that this theme is present in a topic that is taught at the time of the conference: the equation of the circle. This equation is often presented to students as a result to be used, but much less often taught to be understood: Why does the solution set form a circle in the plane? This refers to the

common issue of students at some point understanding "that" and "how", but not understanding "why".

The second theme was related to the Dutch national curriculum change in 2014 that emphasized a shift from attention from rote work to problem-solving: this was coined "mathematical thinking activities", perhaps in contrast to rote exercises that, in the end, do not require much deep thought. So, the teachers want to explicitly address mathematical thinking in their teaching.

The third theme was in the classroom/school level. The lesson took place at the end of 9th grade. At this point, some students have chosen to continue in a science stream in grade 10 and some go to a non-science stream. The differences in mathematical abilities between these students are very large, and this creates a need to approach these two populations in different ways: to differentiate. This is not always easy, and that is why it is addressed by the teachers in the Lesson Study.

Since the teachers could not simply make time in the planning for the students, they decided the Study Lesson would address a very specific part of the usual course. They even specified which specific tasks in the textbook would be replaced by the task in the Study Lesson. Concretely, the mathematical learning goal was, in short, the equation for circles of radius r around the origin: $x^2 + y^2 = r^2$.

After making this decision, the teachers had a first brainstorm about the design of the Study Lesson. An online folder was made for sharing materials and mathematics education (research) articles about these themes and this learning goal. It was agreed that each participant would try and find relevant reading materials and think about ways to teach the subject with attention to the themes. At this point, it becomes clear that another part of the infrastructure is still missing in the Netherlands. There are limited accessible texts available to Dutch teachers on concrete lesson ideas to teach a subject. This contrasts with Japan, where there will be an abundance of reports on lessons on all sorts of subjects available in dedicated journals. Additionally, on the school level, the very busy schedule does not leave the teachers much time for extensive study. This together had made it very difficult for the teachers to find much relevant information on teaching the circle equation or – perhaps more importantly – the more general goals they set themselves. Hence, the brainstorming in the second meeting was mostly based on personal experience of the teachers. To address both mathematical thinking and differences between students, the teachers decided to apply the didactic of guided reinvention (without calling it this way). The problem they decided to set for the students was to reinvent the circle equation in small groups. The guidance was going to be provided by a set of hint cards to support the consecutive steps of the reinvention process. From here the team went on to discuss the details of the lesson plan and hint card. In the end, this led to a successful Study Lesson on a stage observed by almost 150 conference attendants.

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