Teachers' Inquiry in
Mathematics Education

TIME²
Scenarios

Innovative scenarios for inquiry-based mathematics education

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Innovative scenarios for inquiry-based mathematics education

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http://time-project.eu

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Teachers' Inquiry in Mathematics Education

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Introduction

Teachers engage in professional development in various ways. Quite often they are only participants in a lecture or a workshop in which another person tries to inspire them with ideas and experience. It is a fact that this kind of lifelong learning is efficient and appreciated if it also provides support for implementation in the form of ready-made materials and other tips on how to use those resources.

In project TIME, we have set a goal for the whole consortium to investigate how mathematics teachers can improve their own skills in designing innovative teaching and learning situations. We have organized teachers working in partnering schools into nine teams and each team got involved in the complete process of planning, designing, implementing, and reporting new lessons that will help their students master basic mathematical skills. On this quest, they have challenged themselves in terms of creativity, collaboration, didactic and mathematical considerations and, of course, time management. The results comprise the 22 scenarios that are contained in this booklet.

Each scenario specifies the target knowledge, the broader goals and the age of students aimed at by the lesson. Based on these, the teachers have developed didactic materials, designed an intriguing problem for students, and written the outline of the different phases of the lesson. In thinking about and formulating the problem teachers were inspired by Realistic mathematics education. Thus, they have tried to design a problem with a rich context, close to the students' everyday life and current understanding of mathematics, open enough to allow multiple strategies and ways of reasoning and, most importantly, carry the didactic potential to reach the target knowledge.

The structure of the scenario is based on the Theory of didactic situations and follows the template produced in the project MERIA. This implies that each scenario starts with the devolution of the problem from the teacher to the students. In this phase, the students also accept the responsibility for taking autonomous actions in solving the problem which will lead to learning. To deal with many different approaches, the teacher organizes a phase of formulation in which the students present their findings and reasoning. As the goal of inquiry-based education is for the students to discover new knowledge, the next phase, validation, is very important. The lesson should be designed in such a way that the students can not only solve a problem but also be certain if their strategy is optimal. In that way, the students are indeed constructing new knowledge independently and, in that process, develop their confidence in doing mathematics. In the final stage of the lesson, the teacher helps students to connect their own findings with the knowledge shared by the wider (mathematical) community and to link the discovered concept to the standard institutional terminology. This structure guides the students through different stages and mental processes that are all very much important in mathematics education but also supports the teacher in orchestrating lessons that allow students to be creative and think in many diverging directions. The mathematical part of the possible students' solutions is provided at the end of each scenario.

Some scenarios have additional materials (worksheets, tables, presentations, etc.) and all of these are provided in the separate editable file(s) on the project's webpage.

All the scenarios in this booklet have been developed with higher purposes that the teams have set out for themselves as part of engaging in Lesson Study. Hence, all of these activities have been tried out in classrooms multiple times and the experiences are published in the accompanying booklet *TIMEless Practice Reports*. The reports make the rationale behind the scenario transparent and provide many details about the students' reactions, including some advice about the possible changes that can be made. Hence, we strongly advise the readers to use both booklets together and consult both the scenario and the report on a certain topic.

The reader consulting both booklets will notice that the number of scenarios and the number of reports differ. Some of the reports cover two scenarios because they were implemented as consecutive lessons. Furthermore, there are a few scenarios that were produced in the dissemination phase of the project, e.g., for the open lessons at the workshops and the TIME final conference. Finally, the last scenario has been designed and implemented on top of the plan, simply because the team had a very nice idea. Instead of a report, the team invested more time in developing the interactive mobile game and testing it with the students. For the other 18 scenarios, there is a report, and a short overview is also given in the introduction to the other booklet.

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Teachers' Inquiry in
Mathematics Education

Minus and minus gives plus

Making sense for a principle learnt by heart

Team: Midtsjællands Gymnasium, Haslev, Denmark

Target knowledge	A better understanding of negative numbers and multiplication involving negative numbers.
Broader goals	Using symbols and formal mathematical language.
Prerequisite mathematical knowledge	Fundamental arithmetic.
Grade	Age 15-16 (1st grade in Denmark)
Time	60 minutes
Required material	Pen and paper, blackboard, online tools for sharing answers.
Problem: We remember the arithmetical rule from primary school that "minus and minus is plus". What does this actually mean? How would you explain why it is so?	

Phase	Teacher's actions, including instructions	Students' actions and reactions
Devolution (didactical) 5 minutes	The teacher poses small questions to check the students' ability to calculate with negative numbers, for example: calculate $3+(-5), 3\cdot(-5), (-3)\cdot(-5)$. The teacher introduces the main problem and encourages the students in groups of three to come up with a great explanation of why "minus and minus is plus".	Students enter their answers, e.g., using *socrative.com*. Students listen and ask clarifying questions.
Action (adidactical) 10 minutes	The teacher walks around and observes how the groups tackle the challenge. If needed, the students are reminded of the four representations (words, symbols, tables, and graphs), that the class has been focusing on recently.	Students discuss the problem and enter their proposal in *OneNote*, so their considerations are visible to the teacher and the other students.
Formulation (adidactical) 10 minutes	The teacher chooses certain groups to present their work so far.	The representatives of chosen groups present their work, while other students listen and ask questions.
Validation (didactical) 5 minutes	The teacher discusses the different approaches and their potential with the class.	Students listen and engage in the discussion comparing the presentations of different groups.
Action and Formulation (adidactical) 10 minutes	The teacher asks the students to follow a certain idea and develop it further, either their own or an idea of their classmates. The process also reveals some common misunderstandings about negative numbers that the students have from primary school.	Students work as before and try to come up with a better and more precise explanation of the rule. Students present their ideas on posters or the blackboard.

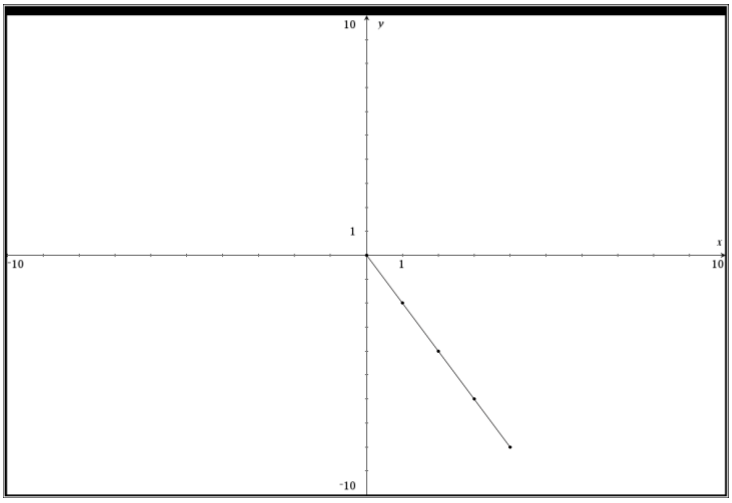
Validation and Institutionalisation (didactical) 10 minutes	The teacher picks up on the different approaches and gives both a geometric explanation and an arithmetic explanation of the problem.	Students listen and take notes.

Possible ways for students to realize target knowledge

- Looking at the function $y = -2x$, we calculate the corresponding values of x and y:

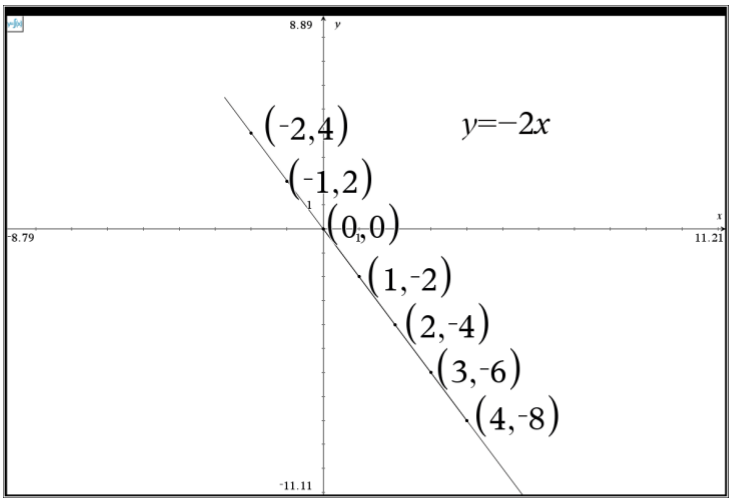
x	0	1	2	3	4
y	0	-2	-4	-6	-8

Plotting these points in a coordinate system and connecting the points with a straight-line yield:



For negative values of x the corresponding value of y "must" be positive for the line to continue "naturally" in the second quadrant. This is a kind of visual explanation showing that

$$(-2) \cdot (-1) = 2, (-2) \cdot (-2) = 4, \text{etc.}$$



- By using a multiplication table, we can argue that the product of two negative numbers must be a positive number. We know how to multiply positive integers, so we start by filling out the bottom right corner:

X	-4	-3	-2	-1	0	1	2	3	4
-4									
-3									
-2									
-1									
0									
1						1	2	3	4
2						2	4	6	8
3						3	6	9	12
4						4	8	12	16

Using pattern recognition, we can work upwards and fill out the rest of the last four columns:

X	-4	-3	-2	-1	0	1	2	3	4
-4						-4	-8	-12	-16
-3						-3	-6	-9	-12
-2						-2	-4	-6	-8
-1						-1	-2	-3	-4
0						0	0	0	0
1						1	2	3	4
2						2	4	6	8
3						3	6	9	12
4						4	8	12	16

Now we can use pattern recognition and work backwards from right to left and fill out all the rows:

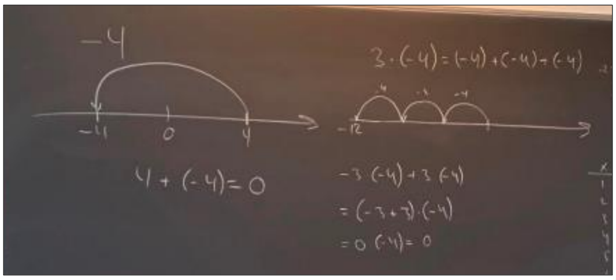
X	-4	-3	-2	-1	0	1	2	3	4
-4	16	12	8	4	0	-4	-8	-12	-16
-3	12	9	6	3	0	-3	-6	-9	-12
-2	8	6	4	2	0	-2	-4	-6	-8
-1	4	3	2	1	0	-1	-2	-3	-4
0	0	0	0	0	0	0	0	0	0
1	-4	-3	-2	-1	0	1	2	3	4
2	-8	-6	-4	-2	0	2	4	6	8
3	-12	-9	-6	-3	0	3	6	9	12
4	-16	-12	-8	-4	0	4	8	12	16

And now the upper left corner shows us, that the multiplication of two positive numbers yields a positive number.

- A negative number is an opposite number of a certain positive number. That is if a is a positive number, then $-a$ satisfies the equation

$$a + (-a) = 0.$$

As an explanation of why the multiplication of two negative numbers is positive, we can give the following example which rests on the law of distribution.



We consider two negative numbers (-3) and (-4) and show that their product is (positive) 12.
By the definition of a negative number, we have that $3 + (-3) = 0$.
By the definition of multiplication, we have that $3 \cdot (-4) = (-4) + (-4) + (-4) = -12$.
By the law of distribution, we have that $(-3) \cdot (-4) + 3 \cdot (-4) = \big((-3) + 3\big) \cdot (-4) = 0 \cdot (-4) = 0$.
So, $3 \cdot (-4) = -12$ is the opposite number of $(-3) \cdot (-4)$ and therefore $(-3) \cdot (-4) = 12$.

It could be that the students are quite confused with this topic and the explanations given in the class. The phrase "minus and minus is plus" can be very ambiguous and it might not be clear that it refers to multiplication. Also, more time might be needed for students to delve into, e.g., the precise definition of a negative number and the distributive law. There should be enough time left for the institutionalisation phase.

Rocket launch

Modelling speed as a motivation for derivatives

Team: Midtsjællands Gymnasium, Haslev, Denmark

Target knowledge	The difference between average and instantaneous speed.
Broader goals	Conceptual understanding of the definition of the differential quotient. Connecting real-life situations and physics to mathematics. Functional modelling from data. Argumentation and use of mathematical language.
Prerequisite mathematical knowledge	Calculating the slope of a line from two points, as well as an elementary understanding of growth and average speed.
Grade	Age 16-17 (2nd grade in Denmark)
Time	60 minutes
Required material	Computer/tablet, calculator, pen and paper. Video clips - where the speed is hidden, but time and altitude are still visible: https://www.youtube.com/watch?v=I3kBg6Uql4w - where the speed is visible: https://www.youtube.com/watch?v=sB_nEtZxPog

Problem:
Consider the video of a rocket launch. What is the average speed of the rocket during the first 60 seconds? Give as precise assessment as you can of the speed of the rocket after precisely 60 seconds. Explain how you calculated the speed.

Phase	**Teacher's actions, including instructions**	**Students' actions and reactions**
Devolution (didactical) 10 minutes	The teacher shows the video to the students (without the speed being visible) and introduces the problem without giving hints. The unit for speed should be mentioned.	Students watch the video and listen to the problem.
Action (adidactical) 25 minutes	The teacher is observing the students' work and considers the order of the following presentations. If some groups are really struggling the teacher may help with the formula of the slope.	Students are working in groups. Every group has a computer since the video must be analysed more closely. Students save their work for the presentation in OneNote.
Formulation (didactical) 15 minutes	The teacher decides the order of the group presentations and invites students to present their findings. Students are encouraged to repeat the arguments of other groups.	The groups take the floor and present their work and their estimation of the speed.
Validation (didactical) 7 minutes	Students' estimations are compared, and strategies are validated by showing the video with the speed being visible.	As the video is shown, it will be discovered which group was the closest to the speed of the rocket.
Institutionalisation (didactical) 3 minutes	The teacher encourages the students to formulate what they have learned about these concepts and what difficulties lie in calculating the speed at an exact point.	Students explain what they have learned and pose clarifying questions. The focus is on the difference between the average speed in an interval and the speed at an exact point in time.

Possible ways for students to realize target knowledge

The lesson is situated at the very start of a period in which the students are being introduced to the subject of differential calculus. The main idea is that students would refer to this lesson later when working with concepts such as the secant line, tangent line, derivative and the notion of limit. If time permits, the secant line and the tangent line can be mentioned by the teacher in the final phase of the lesson.

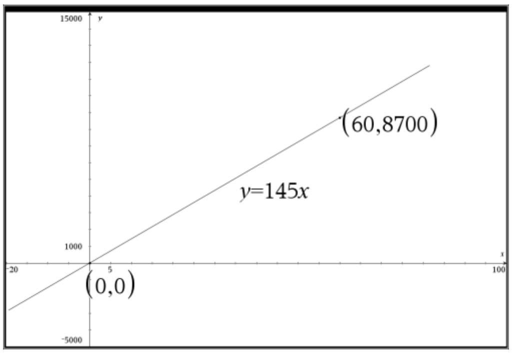
The students may question what kind of altitude is measured in the video and is the direction of the rocket vertical. Concerning measurements, they might get confused about which units to use. One should also note that the precision of the measurements is limited to one decimal.

The average speed of the rocket during the first minute

a) By noticing that after 60 seconds the rocket has reached an altitude of 8.7 km=8700 m, the average speed during the first minute can be calculated as

$$\frac{8700\ m}{60\ s} = \mathbf{145}\ \frac{\boldsymbol{m}}{\boldsymbol{s}} = 145\ \frac{m}{s} \cdot \frac{3600\frac{s}{h}}{1000\frac{m}{km}} = \mathbf{522}\ \frac{\boldsymbol{km}}{\boldsymbol{h}}$$

b) Let $f(t)$ be the function that measures the height in meters of the rocket t seconds after the launch. From the video, we know that $f(0) = 0$ and $f(60) = 8700$. The *slope of the secant line* through these two points is $a = \frac{8700-0}{60-0} = 145$. This is the value of the average speed in $\frac{m}{s}$.



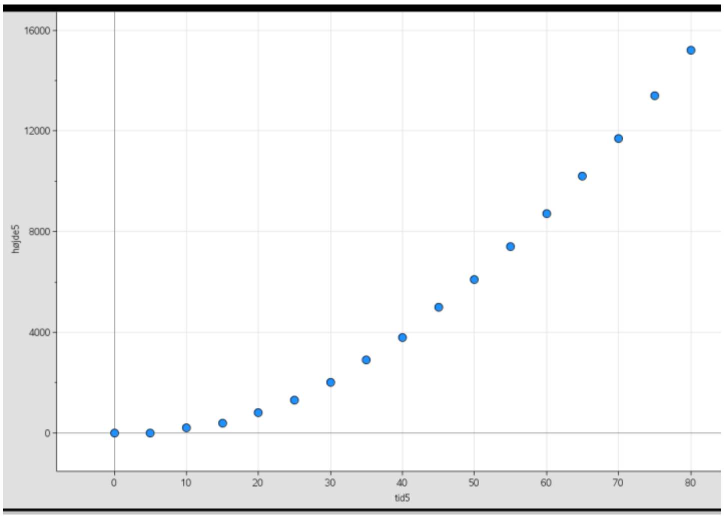
c) One could also use some “black-box”-calculator on the internet, e.g., from the webpage: https://iloebesko.dk/beregn-hastighed-i-km-t-eller-min-km/

Estimate of the instantaneous speed of the rocket after precisely 60 seconds

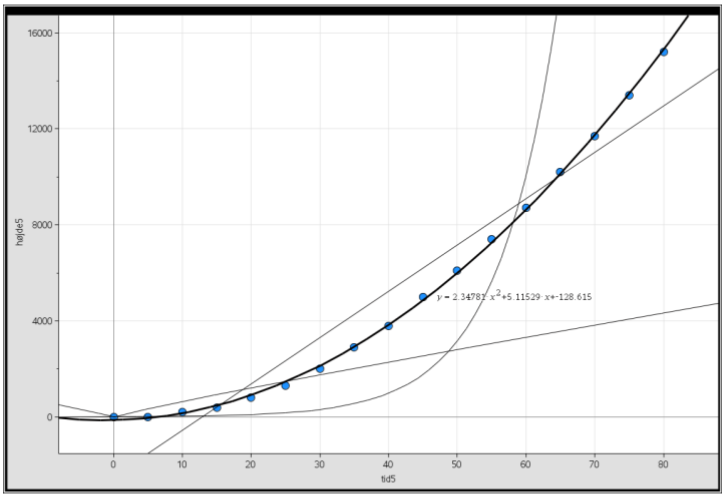
a) Let $f(t)$ be the function that measures the height in meters of the rocket t seconds after the launch. From the video, we gather the information $f(59) = 8400$ and $f(61) = 9000$. The slope of the straight line between these points is $a = \frac{9000-8400}{61-59} = 300$.
Using the average speed in the interval from the 59th second to the 61st second as an estimate, the rocket will travel with a speed of $300\frac{m}{s} = 300\ \frac{m}{s} \cdot \frac{3600\frac{s}{h}}{1000\frac{m}{km}} = \mathbf{1080}\ \frac{\mathbf{km}}{\mathbf{h}}$ exactly one minute after its launch. This is an approximation that can lead in the direction of taking smaller intervals.

b) Analysing the video more systematically and collecting data every five seconds yields the following dataset (left picture). Using the dataset to make a point plot with time (in seconds) on the x-axis and the altitude of the rocket (in meters) on the y-axis yields a graph (right picture).

	A tid5	B højde4	C højde5
=			='højde4*1000
1	0	0	0
2	5	0	0
3	10	0.2	200.
4	15	0.4	400.
5	20	0.8	800.
6	25	1.3	1300.
7	30	2	2000
8	35	2.9	2900.
9	40	3.8	3800.
10	45	5	5000
11	50	6.1	6100.
12	55	7.4	7400.
13	60	8.7	8700.
14	65	10.2	10200.
15	70	11.7	11700.
16	75	13.4	13400.
17	80	15.2	15200.
18			
19			



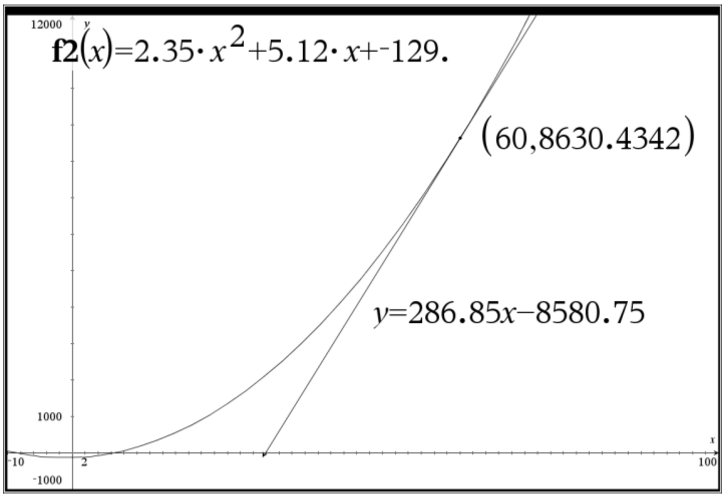
By comparing different types of regression (linear, quadratic, exponential, power), we notice that quadratic regression yields the best model.



So, we use the function

$$f(t) = 2.34781 \cdot t^2 + 5.11529 \cdot t - 128.615$$

as a model of the altitude of the rocket.



By plotting the graph of the function and using the CAS tool to draw the tangent line to the graph at the point $(60, f(60))$ we get the equation $y = 286.85x - 8580.75$ for the tangent line. Thus, the speed of the rocket is $286.85\ \frac{m}{s} = \mathbf{1032.66}\ \frac{\boldsymbol{km}}{\boldsymbol{h}}$ after one minute.

c) Again, using the $f(t) = 2.34781 \cdot t^2 + 5.11529 \cdot t - 128.615$ as a model we obtain, by differential calculus, the speed function

$$v(t) = f'(t) = 2.34781 \cdot 2 \cdot t + 5.11529$$

As $v(60) = 2.34781 \cdot 2 \cdot 60 + 5.11529 = 286.85$ again the speed of the rocket is estimated to be $286.85\ \frac{m}{s} = \mathbf{1032.66}\ \frac{\boldsymbol{km}}{\boldsymbol{h}}$ after one minute.

d) Listening to the commentary of the video, you learn that the rocket is supersonic after approximately 1 minute and 7 seconds. Knowing that the speed of sound is $1236\frac{km}{h}$ we can conclude that the rocket travels at a speed of less than $1236\frac{km}{h}$ after precisely 1 minute. Since the problem focuses on the speed only 7 seconds earlier, the estimate should be quite close.

e) You could argue that there are many inaccuracies in the above mathematical model. In the video, the altitude is given in kilometres with a precision of $0.1\ km$. The timer at the top of the video is given with a precision of 1 second. So, when collecting data from watching the video there will be some given inaccuracies in our dataset.
More importantly, however, one could and should problematize the direction of the rocket which we actually have assumed is vertical. The altitude measured in the video is presumably the vertical altitude of the rocket above the point of launch. So, if the direction of the rocket is not vertical, then this will cause further uncertainties in the calculations.
In the video (where the speed is visible) the speed after 1 minute is measured at approximately $1100\frac{km}{h}$, a bit higher than the calculated estimate. This might partly be explained by the fact that the rocket has travelled further than the given altitude since the trajectory is not vertical.

How many apples are there in a pile?

Using the mean average to make estimates

Team: Midtsjællands Gymnasium, Ringsted, Denmark

Target knowledge	Understanding that the estimate of the average is better if the sample is bigger.
Broader goals	Conversion between different representations of data and functions. Mathematical modelling. Critical thinking.
Prerequisite mathematical knowledge	Average of a finite number of real numbers. Basic operations with fractions.
Grade	Age 15-16 (1st grade in Denmark)
Time	45 minutes
Required material	Datasets with five measurements given in a table.

Problem:
At *Roskilde Market*, there is a booth from *Pi Brewery*, where competition has been set up that involves guessing how many apples are there in the pile knowing that the weight of all apples in the pile is 100 kg. The one who comes closest to the correct number is the winner of a new bike. Information about the apples is given in the following table:

Name	Dennis	Mette	Lotte	Bo	Kaj
Number of apples	5	10	15	20	1
Weight (gram)	1000	1100	2400	2500	100

Phase	Teacher's actions, including instructions	Students' actions and reactions
Devolution (didactical) 5 minutes	The teacher presents the problem to the students.	Students listen and ask clarifying questions.
Action (adidactical) 15 minutes	The teacher circulates in the classroom and observes the students' approaches. (S)he thinks about the order of presentations based on the strategies that emerge in students' work.	Students work with the dataset and prepare presentations of their ideas on a poster. See "*Possible ways for students to realize target knowledge*".
Formulation (adidactical) 10 minutes	The teacher invites the students to present in the order based on observations.	If several groups do the same thing with the dataset, each group should only add new information that was not yet presented to avoid repetition.
Validation (didactical) 10 minutes	The teacher asks the students to compare different approaches and explain which one they find better. The teacher may also challenge some of the approaches.	It is expected that the students comment that the size of the sample is important. Some might be confused with the difference between the average of all and the average of averages.
Institutionalisation (didactical) 5 minutes	The teacher emphasizes the approaches based on larger samples and underlines the importance of statistically correct reasoning.	Students write notes about the approaches that were new to them and follow the conclusion of the lesson.

Possible ways for students to realize target knowledge
The main idea is to calculate the number of apples by dividing the total mass of 100 kg by the "average mass of an apple". The approaches differ in the ways how to interpret the "average mass of an apple". Some of the approaches are more in line with the statistical way of thinking and the purpose of the lesson is for the students to realize that during the discussion.

1. Using the information from the table that Kaj took 1 apple with a weight of 100 grams to represent all the apples you get the answer

$$\text{number of apples} = \frac{100\ \text{kg}}{0.1\,\frac{\text{kg}}{\text{apple}}} = 1000\ \text{apples}.$$

2. The students find the mass of an apple based on the mass of the 20 apples. From here we find the number of apples in the pile weighing 100 kg. Students who think this is a better estimate will calculate:

$$\text{weight of 1 apple} = \frac{2500\ \text{g}}{20} = 125\ \text{g},$$

$$\text{number of apples} = \frac{100\ \text{kg}}{0.125\,\frac{\text{kg}}{\text{apple}}} = 800\ \text{apples}.$$

Compared to the previous approach, in this approach students show an understanding that apples will differ and that using the mean average will lead to a better estimate than using only one random apple.

3. Students calculate the average of an apple by finding the average of 1, 5, 10, 15 and 20 apples. Then they take the average of the previously calculated averages. From here, the number of apples in the pile is found by dividing the average mass of an apple up to 100 kg

1 apple	5 apples	10 apples	15 apples	20 apples
100 g	1000 g	1100 g	2400 g	2500 g
100 g/apple	200 g/apple	110 g/apple	160 g/apple	125 g/apple

The average of these 5 averages is:

$$\frac{100 + 200 + 110 + 160 + 125}{5} = 139\,\frac{\text{g}}{\text{apple}}$$

$$\text{number of apples} = \frac{100\ \text{kg}}{0.139\,\frac{\text{kg}}{\text{apple}}} \approx 719\ \text{apples}.$$

In this approach, the students show a typical misconception that the average of the averages for each subsample will be a good estimate of the average of all the apples in the sample.

4. Students find the average mass of an apple by averaging the masses of all the apples for which the data are given in the table. The total number of apples in the table is 51, so the average mass of one apple is calculated as:

$$\text{weight of 1 apple} = \frac{m_1 + m_5 + m_{10} + m_{15} + m_{20}}{51}$$
$$= \frac{100\ \text{g} + 1000\ \text{g} + 1100\ \text{g} + 2400\ \text{g} + 2500\ \text{g}}{51}$$

$$= \frac{7100\text{ g}}{51} = 139.22\text{ g}$$

The number of apples in the pile of 100 kg is then:

$$\text{number of apples} = \frac{100\text{ kg}}{0{,}13922\,\frac{\text{kg}}{\text{apple}}} \approx 718\text{ apples.}$$

In this approach, students show awareness that the dataset consists of similar information that is not to be compared but used altogether. Indeed, this is a better strategy than the previous one because in the previous it might be very problematic to treat each subsample as if it is equally important (i.e., big).

5. There could be many different approaches based on the graphic representation of the data in the coordinate system. It is an interesting idea that might lead to the recollection of the formula for the slope of the line passing through two given points, or it might even open a valuable idea of using linear regression, but **it should be emphasized that the data in the dataset do not represent a dependency of two variables and the different inputs have different weights (importance), so linear regression is not an appropriate method.** Despite this, we show the possible results that students might get this way.

   a) Students select two random points and calculate the slope using the two-point formula
   $$a = \frac{y_2 - y_1}{x_2 - x_1}$$
   where (x_1, y_1) and (x_2, y_2) are two points on a straight line.
   Choosing, e.g., (1,100) and (20,2500)
   $$a = \frac{2500 - 100}{20 - 1} = 126.3$$
   From here we get the answer:
   $$\text{number of apples} = \frac{100\text{ kg}}{0{,}1263\,\frac{\text{kg}}{\text{apple}}} \approx 792\text{ apples.}$$

   In this approach, the students assume that the two pairs of data represent the dependency of two variables, neglecting that the different pairs represent different samples of different sizes and nothing more. This approach can be challenged by observing that the same procedure for the pairs (1,100) and (5, 1000) would yield ≈ 445 apples.

   If the students select (0,0) and one of the other points in the table and determine the slope using a two-point formula, they are actually working as in the second point. A graphical representation might only lead the students to choose the first or the last point.

   b) Students calculate slopes using a two-point formula based on several or all the points and try to see if there is a pattern. For example, they could make the following list of calculations.

   The slope based on (1,100) and (5,1000) is $a = \frac{1000-100}{5-1} = 225.$
   The slope based on (5,1000) and (10,1100) is $a = \frac{1100-1000}{10-5} = 20.$
   The slope based on (10,1100) and (15,2400) is $a = \frac{2400-1100}{15-10} = 260.$
   And finally, the slope based on (15,2400) and (20,2500) is $a = \frac{2500-2400}{20-15} = 20.$

The average slope is calculated as

$$\frac{225+20+260+20}{4}=131.25.$$

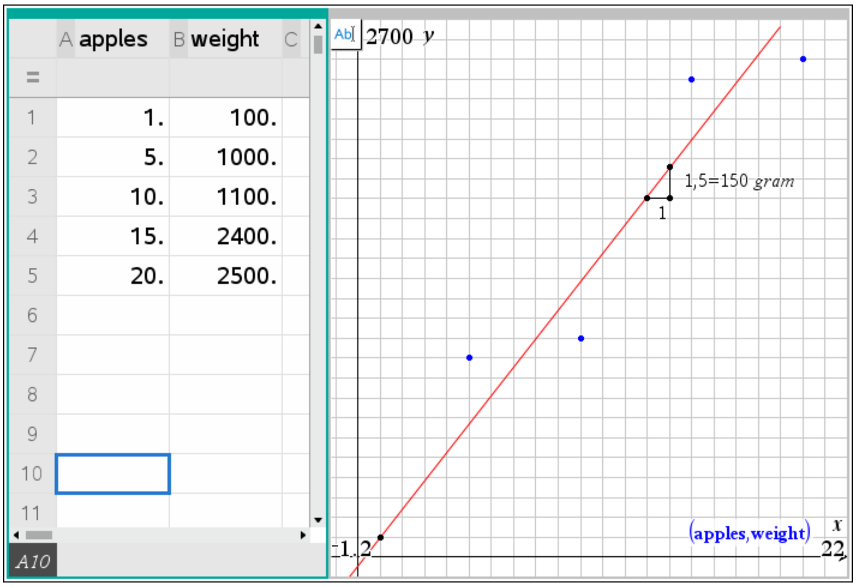
Again, we can calculate the estimate of the number of apples in the pile as:

$$\text{number of apples}=\frac{100\text{ kg}}{0{,}13125\,\frac{\text{kg}}{\text{apple}}}\approx 762\text{ apples.}$$

Again, the same misconception arises that it makes sense to compare the data given in the table as if each entry is equally important.

c) Students select (0,0) and one of the other points in the table and determine the slope using a two-point formula. Here, the students might choose the first or the last or any other point. For example, if (20,2500) is chosen as a second point, these students would obtain the same solution as the students taking the average as seen in the table earlier.

d) Students draw a line through the points so that they are equally distributed above and below the line. They determine the slope and the constant term graphically. See the illustration below (here the students have made this by hand).



Finding the optimal line

Discovering the method of linear regression

Team: Midtsjællands Gymnasium, Ringsted, Denmark

Target knowledge	Discovering the method of linear regression as the model.
Broader goals	Conversion between different representations of data and functions. Mathematical modelling. Critical thinking.
Prerequisite mathematical knowledge	Scatter plots. The equation of a line through two given points.
Grade	Age 15-16 (1st grade in Denmark)
Time	45 minutes
Required material	Datasets with five measurements given in a table. Size A3 mm paper, one per group with the group number. Magnets.

Problem:
Consider the following set of data. Your task is to determine the line that would fit the data in some way that you consider optimal. Find that line and justify your choice.

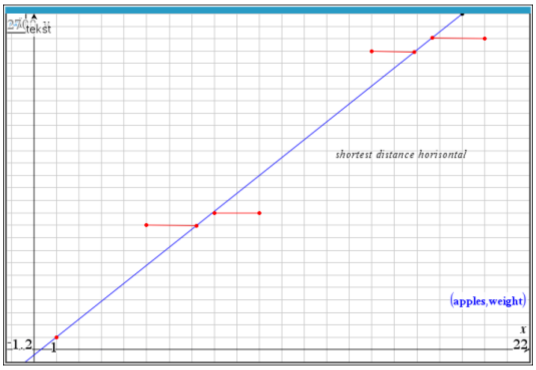
Measurements	1	2	3	4	5
Values of X	5	10	15	20	1
Values of Y	1000	1100	2400	2500	100

Phase	**Teacher's actions, including instructions**	**Students' actions and reactions**
Devolution (didactical) 5 minutes	The teacher presents the problem: How to determine the optimal line while considering all the data points? The teacher shares A3 mm paper and asks the students to present their solutions on that paper.	Students listen and try to understand the problem. Maybe ask clarifying questions.
Action (adidactical) 12 minutes	The teacher circulates among the students and observes the students' different approaches. If a group is not at all able to get the points drawn into the coordinate system after a few minutes, then they are given a sheet where the points are drawn, so they just need to relate it to the optimal line through the points.	Students try to solve the problem. They could consider lines based on different approaches: - Horizontal distance - Triangles - Perpendicular distance - Circles - Vertical distance - Least squares
Formulation (didactical) 5 minutes	The teacher invites groups with different strategies to come to the board and show their solutions.	The selected groups present their solution to the problem, while the other students listen and ask critically about the solutions.
Validation (didactical) 5 minutes	The teacher directs the class discussion about the optimal linear model towards the method of linear regression by pointing out the parts of the method appearing in the students' strategies.	Students should discuss different strategies and their advantages. Following the teacher's instructions, they should progress towards the conclusion to use the method of least squares.

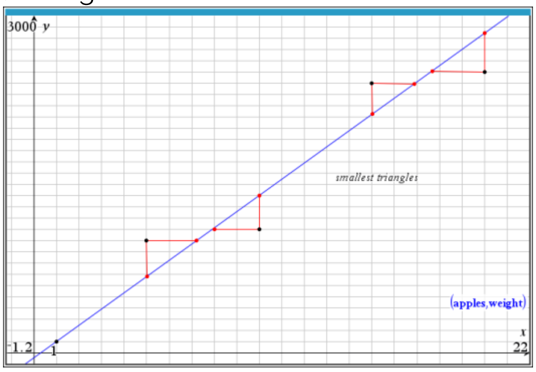
Institutionalisation (didactical) 5 minutes	The teacher summarizes the discussion and formally presents the concept of the least squares method.	Students write notes about what they have learned.

Possible ways for students to realize target knowledge
The optimal line could be, as shown earlier, the line for which we have taken into consideration the same number of points on both sides of the line. Some examples are shown in the following pictures.

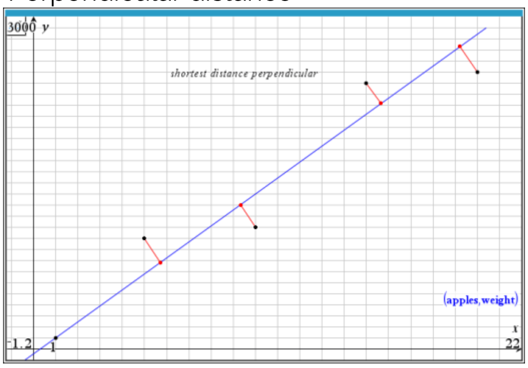
a) Horizontal distance



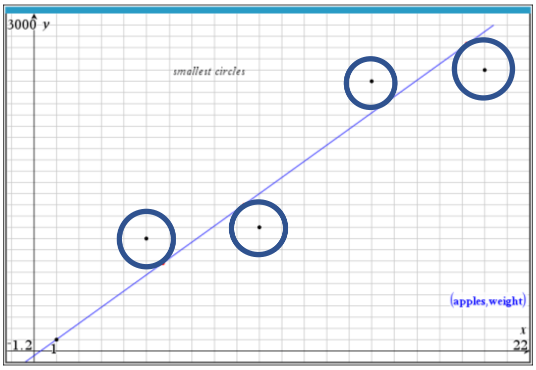
b) Triangles



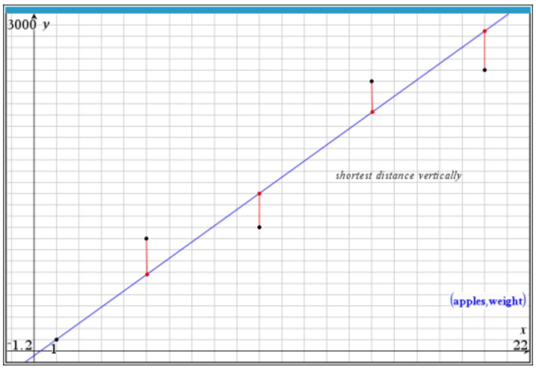
c) Perpendicular distance



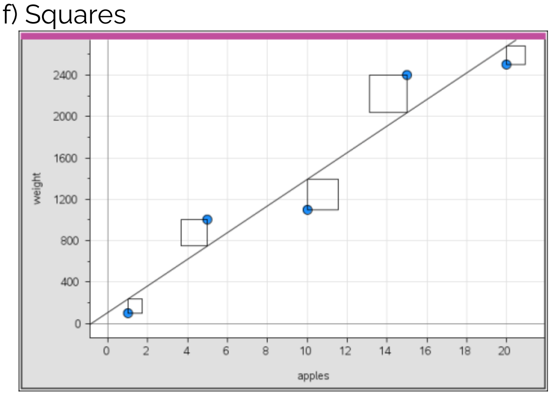
d) Circles



e) Vertical distance



f) Squares



The main discussion might be about the reasons we look at the vertical distance and not the perpendicular. The students might also discuss what are dependant and independent variables and what are they allowed to choose freely and what is given. A further study could be to look at the least squares method and include the deviation of the residuals. This can later form the basis for talking about confidence intervals for the slope when the residuals can be assumed to be normally distributed.

Ringsted Hill

Constructing a piecewise linear function

Team: Midtsjællands Gymnasium, Ringsted, Denmark

Target knowledge	Producing a piecewise-defined function and describing it with a function expression.
Broader goals	Recognizing and describing a graph using elementary functions. Translating a design into a mathematical expression using mathematical language. Getting the intuition about continuous and differentiable functions.
Prerequisite mathematical knowledge	Expressions and graphs of some elementary functions. Using Ti-Nspire (or other software) to define a "slider" and use it in problem-solving.
Grade	Age 15-17 (1st or 2nd grade in Denmark)
Time	60 minutes
Required material	Computers with access to the needed software. A pre-constructed coordinate system (Size A3). Marker pens in two colours. A toolbox with a catalogue of the fundamental functions and their graphs and also guides for using sliders in Ti-Nspire and other tips and tricks.

Problem:
The company Curves for Everyone has been commissioned to make a parking garage in the buildings at the old mill in Ringsted. Inspired by Copen Hill (figure to the right), there will be a new attraction in Ringsted, namely a ski slope.

The building is 35 m high, and the hill extends 45 m from the wall of the parking garage.
How do you think this Ringsted Hill should be designed? Use mathematical formulas to describe your design.

Phase	Teacher's actions, including instructions	Students' actions and reactions
Devolution (didactical) 5 minutes	The teacher hands over the problem to the students and tells them what tools they have available.	Students listen and ask clarifying questions.
Action (adidactical) 25 minutes	The teacher notes the students' approaches and plans the formulation phase (in order of increasing complexity).	Students are working on their designs. The possible approaches are presented below.
Formulation (didactical/ adidactical) 15 minutes	The teacher has organized the board, so the groups know where to place their designs (magnets are used to hang posters).	The groups present their designs to the class, while the other students listen and ask questions.
Validation (didactical/ adidactical) 10 minutes	A discussion on how to define functions in different intervals and eventually how to achieve that the final function is continuous.	Students bring up ideas to write down the expression of the final (piecewise defined) function.
Institutionalisation (didactical) 5 minutes	The teacher makes a summary of the students' functions and an introduction to the notation for a piecewise function	Students listen and note the significance of the notation for a piecewise-defined function and ask clarifying questions.

	$f(x) = \begin{cases} f_1(x), \ x \in I_1 \\ f_2(x), \ x \in I_2 \\ f_3(x), \ x \in I_3 \end{cases}$ using one of the expressions from the groups as an example. We might show how to do it in Ti-Nspire.	

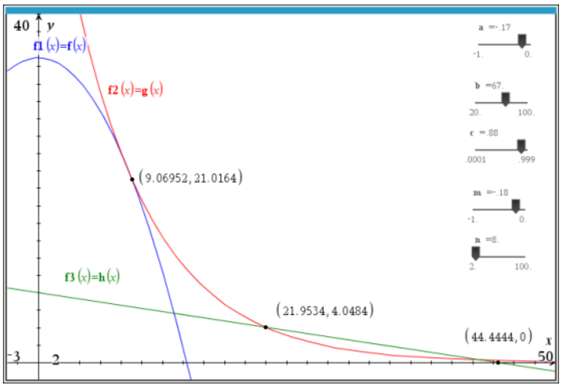
Possible ways for students to realize target knowledge

It seems that the design does not depend on the mathematical level of the students, but more on their experience of skiing. The use of a mathematical language, on the other hand, depends on their mathematical level. Students with a low degree of abstraction may be able to use their CAS tools in an advanced way, so we need to think about the students' approaches both in a mathematical and in a CAS-oriented way.

We will not show all the cases from the table here but will just give you some examples of what the use of CAS could be like. If their abilities to use Ti-Nspire are poor, then giving them instructions is useless because they spend too much time trying to program Ti-Nspire instead of doing math. Teachers can certainly then imagine the pen-and-paper approaches.

See the example below. Students might be missing a certain part of the graph, hence designing a graph that is just a decreasing linear or exponential function.

- Example of D5 in the table



We start the design of the hill by defining a polynomial expression with a slider for the coefficient a. Here we will only look at negative values and the constant term must be 35 because this was the height of the building. And finally we put the vertex on the y−axis.

$f(x):=a \cdot x^2+35$ ▸ *Udført*

The next piece will be that of an exponential function. Here we know 0<c<1 because it should be a decreasing function, and we but the coefficient b>0 in an appropriate interval perhaps trying some different max−values, if we are not satisfied with our slope.

$g(x):=b \cdot c^x$ ▸ *Udført*

Finally we will end up with a linear expression and here we want only negative numbers for the slope m and also small numbers, because we need a more flat ending. The constant n are changed if necessary

$h(x):=m \cdot x+n$ ▸ *Udført*

Once satisfied, we find the intersections using the Intersect tool to find the intervals and the expressions can be read on the sliders.

- Example of D4 in the table:

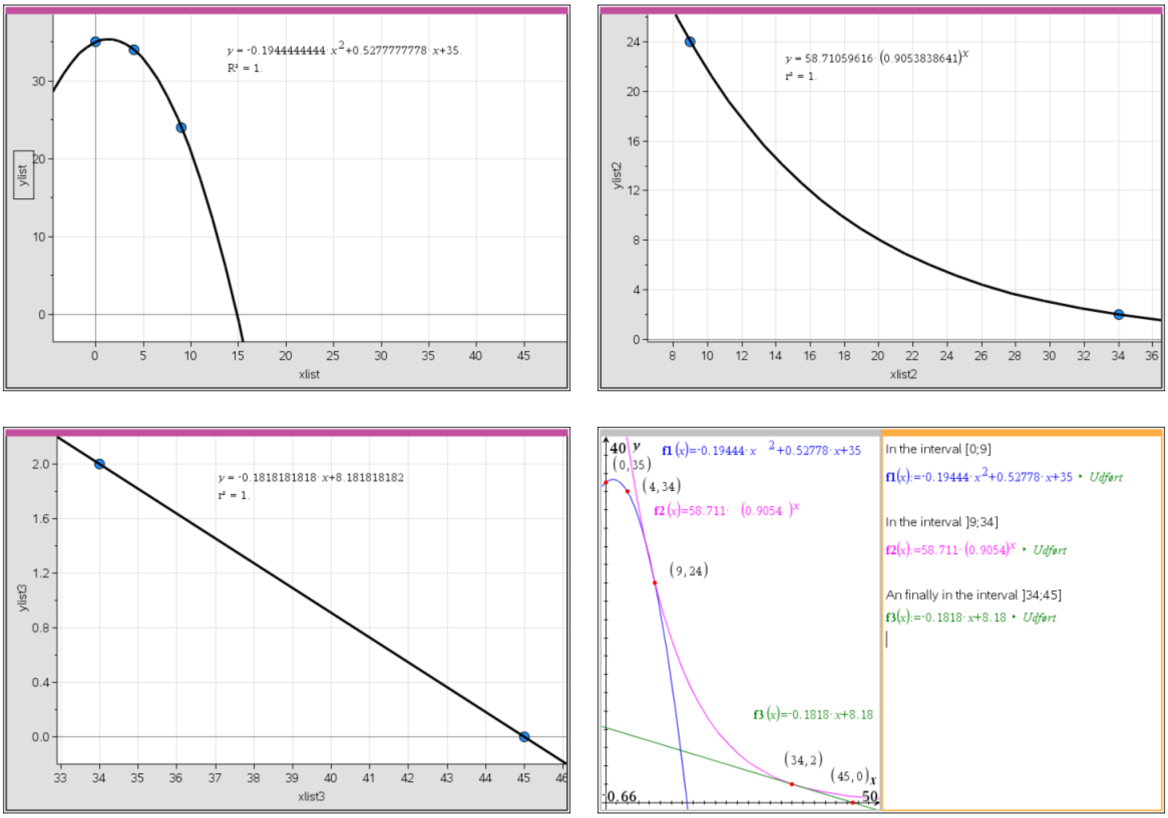
	A xlist	B ylist	C xlist2	D ylist2	E xlist3	F ylist3	G
=							
1	0.	35.	9.	24.	34.	2.	
2	4.	34.	34.	2.	45.	0.	
3	9.	24.					
4							

B ylist

Here we have started plotting points in the graph window and decided what kind of function we wanted and also how many points to use for each type of function.

Then we made 3 pairs of lists to do the regressions, the first set of lists being for a polynomial expression, the second set of lists for an exponential, and finally the third for a linear function.

To see the result we defined our three functions to draw the graphs in the first graph window



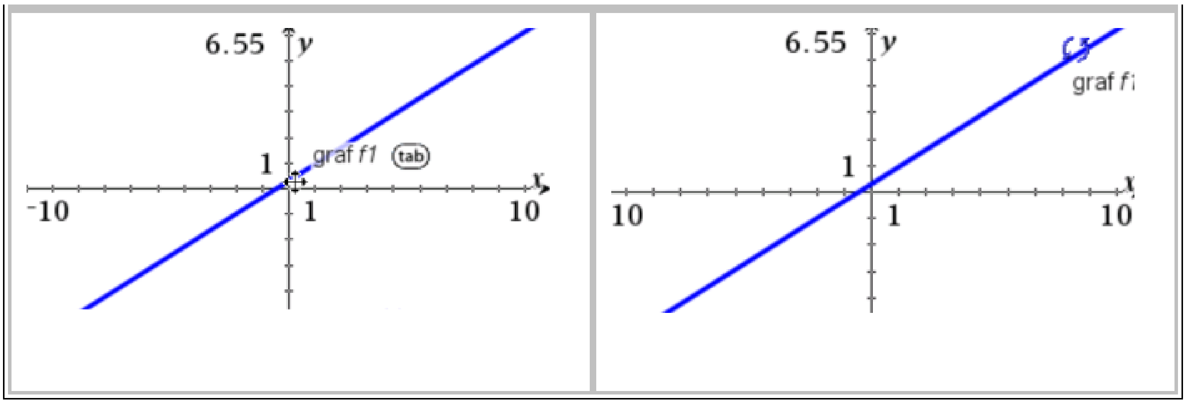
This last example could be done as well by the students with pen and paper using two-point formulas for the linear and exponential functions and perhaps solving three equations with three unknowns with some help from their CAS tool for the polynomial expression.

NOTE: See the project's webpage for the following editable materials in Word.

STUDENTS GUIDE TO TI-NSPIRE

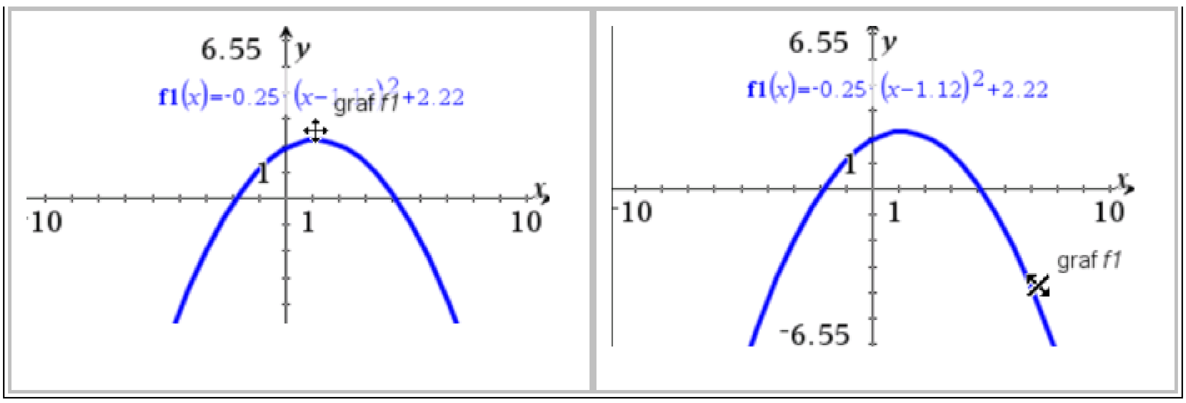
- ***Vertical movement and changing the slope - Linear functions***

1) Open the application Graph
2) Draw the graph for the function $f1(x) = x$.
3) **Vertical movement:** Place the cursor in the middle of the line (note the cross icon) and drag up or down to move the line
4) **Change the slope:** Place the cursor at the end of the line (note the circle icon) and drag it up or down to change the slope of the line.



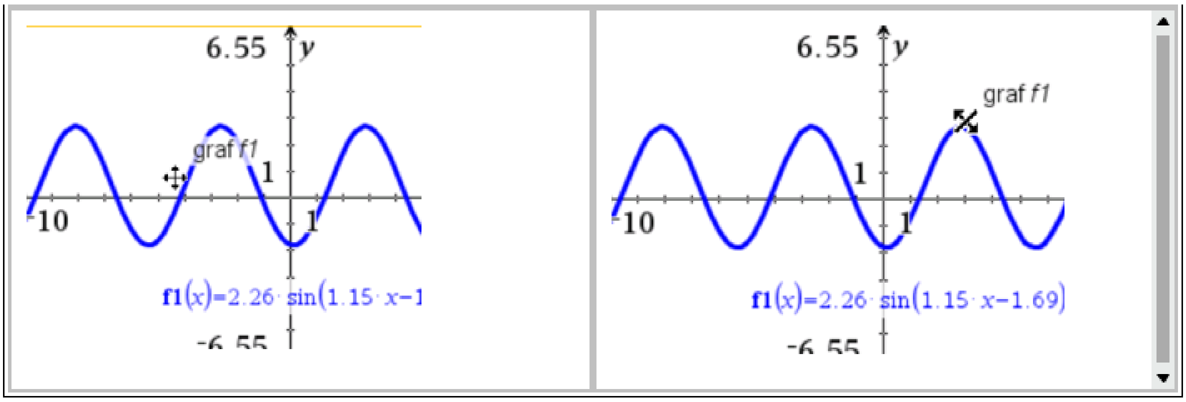
- ***Vertical movement and changing the parabola***

1) Open the application Graph
2) Draw the graph for the function $f1(x) = x^2$
3) **Vertical movement:** Place the cursor in the middle of the parabola (note the cross icon) and drag up or down to move the graph
4) **Changing the direction or the distance between the branches of the parabola:** Place the cursor at the end of one of the branches (note the icon line-and-arrows) and change by dragging up or down



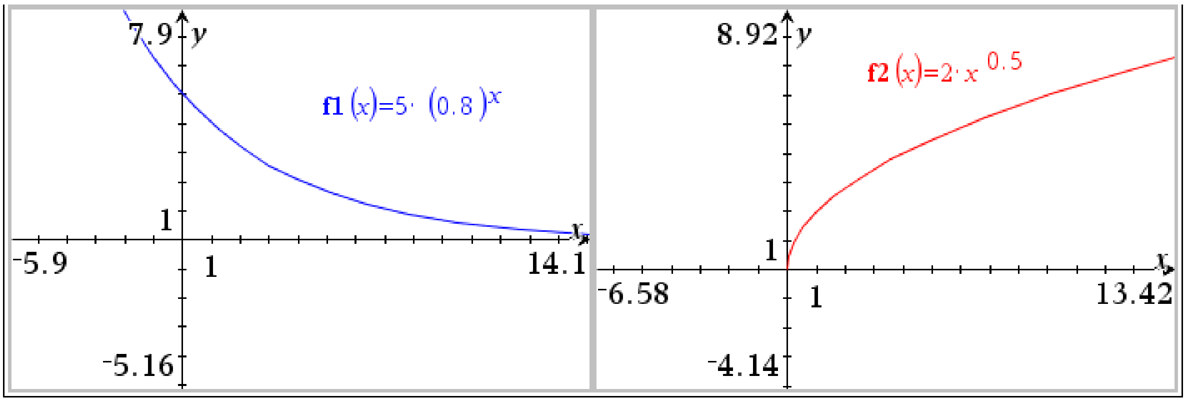
- ***Changing amplitude and period for trigonometric functions***

1) Open the application Graph
2) Draw the graph for the function $f1(x) = \sin(x)$
3) **Vertical movement:** Place the cursor in the middle of the graph (note the cross icon) and drag up or down to move the graph
4) **Changing the amplitude and period:** Place the cursor by an extremum (minimum or maximum) and drag it up or down (amplitude) or horizontally (period). Note the line-and-arrows icon.



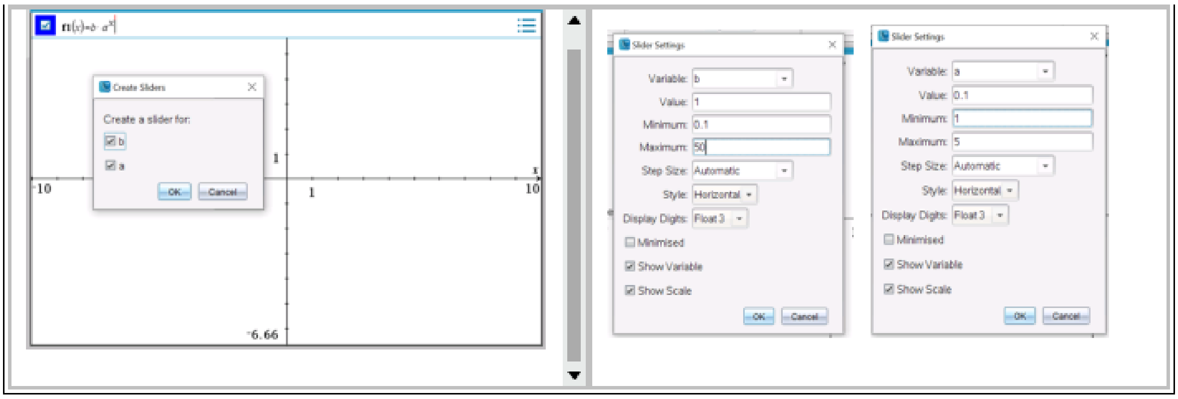
- ***Elementary functions where it is not possible to change the graph by dragging it***

1) Exponential functions: $f1(x) = b \cdot a^x$
2) Power functions: $f2(x) = b \cdot x^a$
3) Instead, you can use sliders. See below.



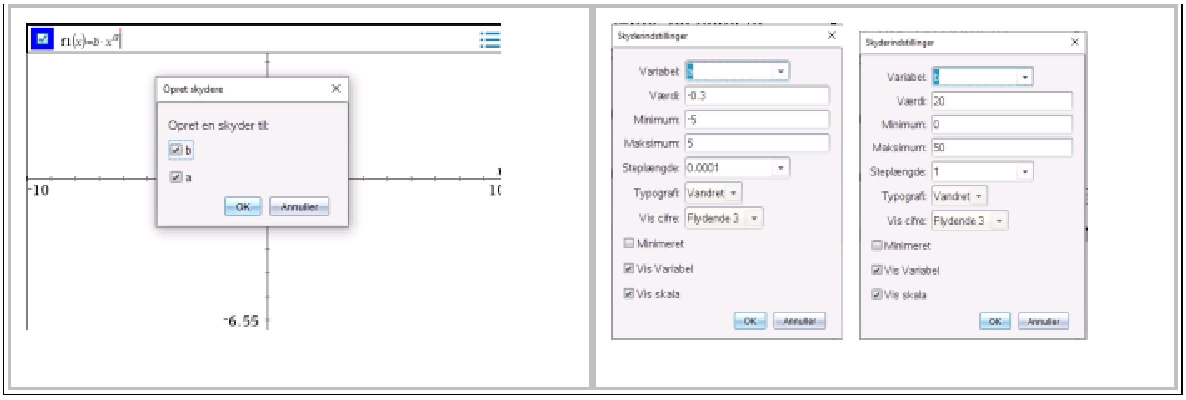
- ***Sliders for exponential functions***

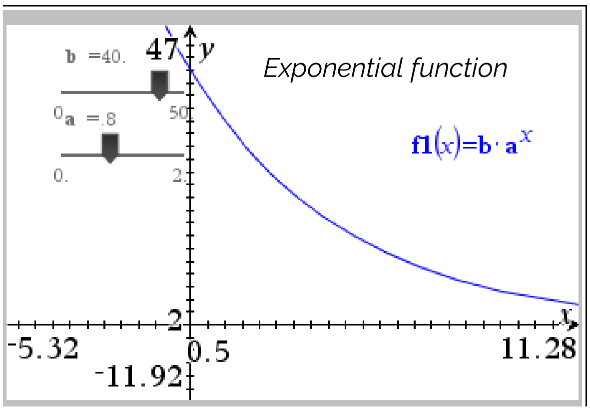
4) Open the application Graph
5) Draw the graph of the function $f1(x) = b \cdot a^x$ and create sliders for the constants a and b
6) Make a right-click on the a-slider and select "Settings". Adjust the settings for the constant a.
7) Make a right-click on the b-slider and select "Settings". Adjust the settings for the constant b.

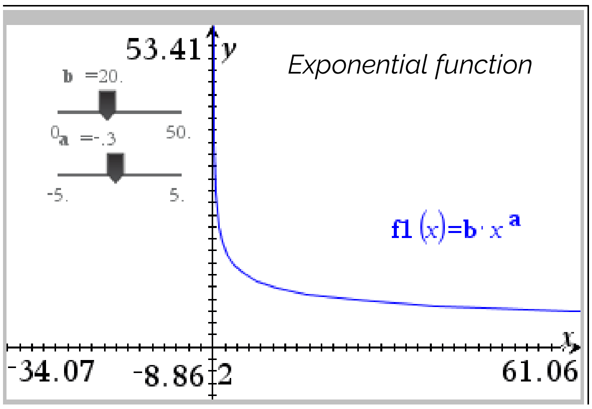


- ***Sliders for power functions***

1) Open the application Graph
2) Draw the graph of the function $f1(x) = b \cdot x^a$ and create sliders for the constants a and b
3) Make a right-click on the a-slider and select "Settings". Adjust the settings for the constant a.
4) Make a right-click on the b-slider and select "Settings". Adjust the settings for the constant b.





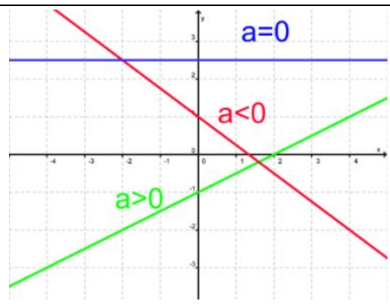


CATALOGUE OF FUNCTIONS

Linear functions:

The form: $f(x) = ax + b, \;\; x \in \mathbb{R}$

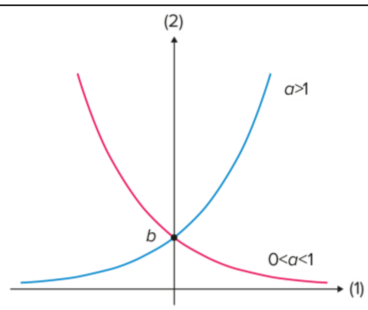
The constants $a, b \in \mathbb{R}$
The constant a is called the slope.
The constant b is called the constant term.



Exponential functions:

The form: $f(x) = b \cdot a^x, \;\; x \in \mathbb{R}$

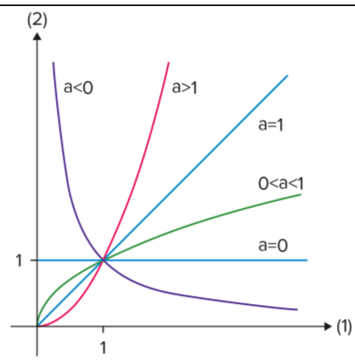
The constants $a, b \in \mathbb{R}_+$ and $a \neq 1$.
The constant a is called the base of the function.
The constant b is called the starting value



Power functions:

The form: $f(x) = b \cdot x^a, \;\; x \in \mathbb{R}_+$

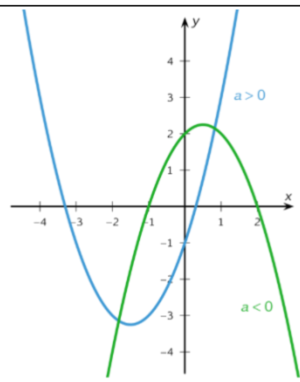
The constants $a \in \mathbb{R}$ and $b \in \mathbb{R}_+$.
The constant a is called the exponent of the function.
The constant b is called the constant of proportionality.

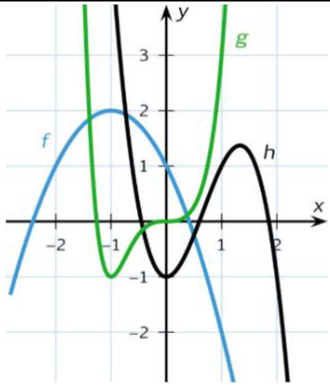


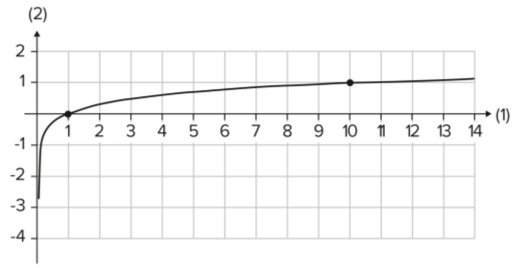
2nd-degree polynomials:

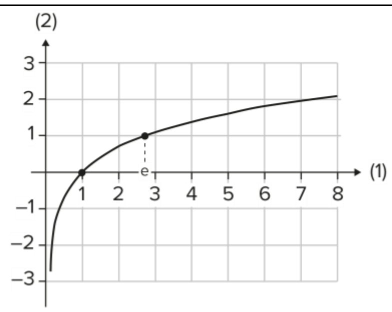
The form: $f(x) = ax^2 + bx + c, \;\; x \in \mathbb{R}$

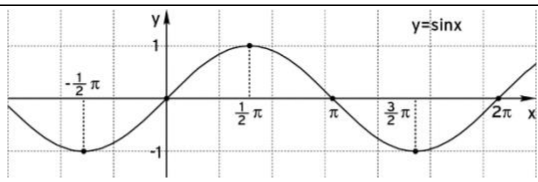
The constants $a, b, c \in \mathbb{R}$ and $a \neq 0$.
The constants a, b and c are called coefficients.

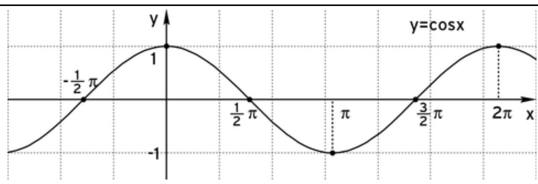


Polynomials:	
The form: $f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 x, \ \ x \in \mathbb{R}$ The constants $a_n, a_{n-1}, \ldots, a_1, a_0 \in \mathbb{R}$ and $a_n \neq 0$ The constants $a_n, a_{n-1}, \ldots, a_1$ and a_0 are called coefficients.	

Decimal logarithm (logarithm with base 10):	
The form: $f(x) = \log(x), \ \ x \in \mathbb{R}_+$	

The natural logarithm:	
The form: $f(x) = \ln(x), \ \ x \in \mathbb{R}_+$	

The sine function:	
The form: $f(x) = \sin(x), \ \ x \in \mathbb{R}$	

The cosine function:	
The form: $f(x) = \cos(x), \ \ x \in \mathbb{R}$	

Harmonic oscillations:	
The form: $f(x) = a \cdot \sin(bx + c) + d, \quad x \in \mathbb{R}$ The constants $a, b \in \mathbb{R}_+$ and $c, d \in \mathbb{R}$ The constant a is called the amplitude. The constant b is called the frequency of the oscillation. The constant c is called the phase shift. The constant d is called the constant of the vertical displacement.	

Ringsted Campus Festival

Optimization of a rectangle area using a quadratic function

Team: Midtsjællands Gymnasium, Ringsted, Denmark

Target knowledge	Modelling the area of a rectangle as a quadratic function of the perimeter and one side and finding the maximum of such function.
Broader goals	Argumentation / mathematical reasoning, modelling and translation of problems formulated in a natural language into a mathematical language. Motivating students for calculus based on a realistic problem.
Prerequisite mathematical knowledge	Area formula for the rectangle. Scatter plots. Graphing the quadratic function. Concept of a maximum of a function and the formula for the vertex of the graph of a quadratic function.
Grade	Age 16-17 (2nd grade in Denmark)
Time	60 minutes
Required material	A non-elastic cord with a length of 205 cm (or other lengths). A3 posters for presentations and markers. Rulers. Magnets.

Problem:

After Covid-19, a group in Ringsted municipality has decided to arrange a festival for young people. They have hired a company to frame the festival area. The company visits all high schools and other schools for young people to borrow a fence to frame the festival area. They end up having 205 m of fences.

The fence will form the area shown in Figure 2. Three sides will be framed using the fence and the fourth side will be the stage area.

Come up with some arguments on how to dimension the festival area so that we can accommodate as many participants as possible.

Figure 1: fence for the festival area

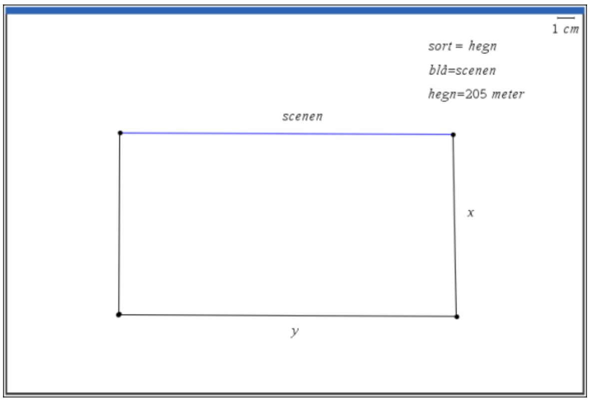


Figure 2: model of the festival area

Translation from Danish
Sort: black
Hegn: fence
Blå: blue
Scenen: the stage

Phase	Teacher's actions, including instructions	Students' actions and reactions
Devolution 1 (didactical) 5 minutes	The teacher asks the students to determine some possible lengths, widths and areas related to the problem. If necessary, (s)he reminds the students of how to calculate the area.	Students listen and ask clarifying questions. Students must have some concrete examples of how to determine the area of a rectangle because the rest of the work depends on the understanding of which figure to work with and how the area is determined.
Action (adidactical) 3 minutes	The teacher organizes the board ready for the students' input.	Students find areas based on length and width they decide for themselves.
Formulation (adidactical) 2 minutes	The board is organized in a table for width, length, and area.	Students write a list of possible areas on the board.
Validation (didactical) 5 minutes	The teacher organizes a short discussion on the validity of dimensions – do the lengths and widths fit with the 205 cm?	Students might see if there are some wrong solutions and what is the problem. Also, they see different possible areas.
Institutionalisation (didactical) 2 minutes	Concluding that the relationship between length and width is $length + 2 \cdot width = 205$	Students listen and take notes.
Devolution 2 (didactical) 3 minutes	The main problem is now presented to determine the maximum area with an emphasis on the argumentation of the optimal solution.	Students listen and ask clarifying questions.
Action (adidactical) 13 minutes	The teacher walks around listening and planning the subsequent formulation phase. Students are given A3 paper and markers to make presentations.	Students use paper and pencil or technology to deal with the data and develop strategies for solving the problem. See *Possible ways for students to realize target knowledge* below.
Formulation (didactical/ adidactical) 10 minutes	The teacher orchestrates the presentations starting from the groups that had more concrete approaches towards more elaborate explanations.	Students take turns presenting their suggestions on how the length and width should be chosen.
Validation (didactical/ adidactical) 10 minutes	The teacher directs the focus of the class to the arguments rather than the maximum (just a number).	Students discuss different presentations. If the answers are very different, they compare them. Students that had more elaborate explanations might answer additional questions.
Institutionalisation (didactical) 5 minutes	Following up on the students' presentations, the teacher shows the general function expression, and its graph and calculates the maximum as the vertex of the graph. The lesson ends with the announcement that by learning calculus we will be able to determine maxima for other types of functions.	Students take notes.

Possible ways for students to realize target knowledge

Based on our experience it is crucial for the students to focus on the relationship between the length and the width, which is seen by the two devolutions in the scenario. The first devolution is planned to support the students in understanding the problem. In the second devolution, the emphasis is on the argumentation of the optimal solution in order to push students further from trial-and-error solutions.

- Some students may use the string to make concrete models of the rectangle and measure the lengths and widths, different results can be written in a table:

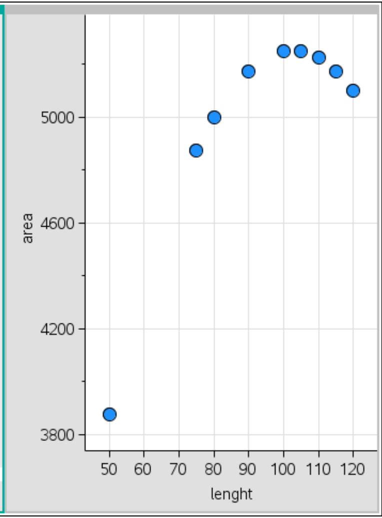
  If the length is $100\ m$*, the width will be* $\frac{205-100}{2} = 52.5$ *giving the area* $100 \cdot 52.5 = 5250\ m^2$.
  If the length is $80\ m$*, the width will be* $\frac{205-80}{2} = 62.5$ *giving the area* $80 \cdot 62.5 = 5000\ m^2$.
  If the length is $105\ m$*, the width will be* $\frac{205-105}{2} = 50$ *giving the area* $105 \cdot 50 = 5250\ m^2$.
  If the length is $103\ m$*, the width will be* $\frac{205-103}{2} = 51$ *giving the area* $103 \cdot 51 = 5253\ m^2$.

  The answer will be $5250\ m^2$ by trying to begin with a length of 100 meters.

- After obtaining a list of some possible dimensions, students insert the points in a spreadsheet and make a point plot. From here, they guess ahead by seeing where a maximum value will be and for which x.

	A lenght	B width	C area	D
=			=lenght*width	
1	50.	77.5	3875.	
2	80.	62.5	5000.	
3	100.	52.5	5250.	
4	110.	47.5	5225.	
5	120.	42.5	5100.	
6	105.	50.	5250.	
7	90.	57.5	5175.	
8	115.	45.	5175.	
9	75.	65.	4875.	
10				
11				

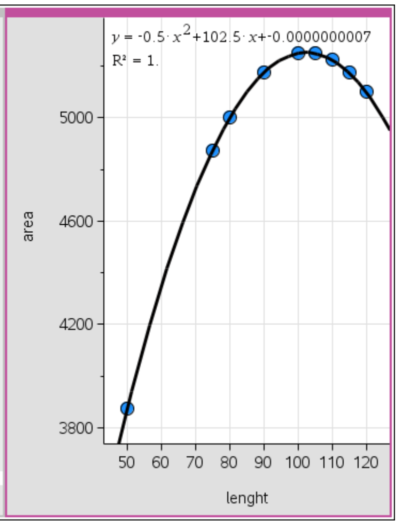
B10



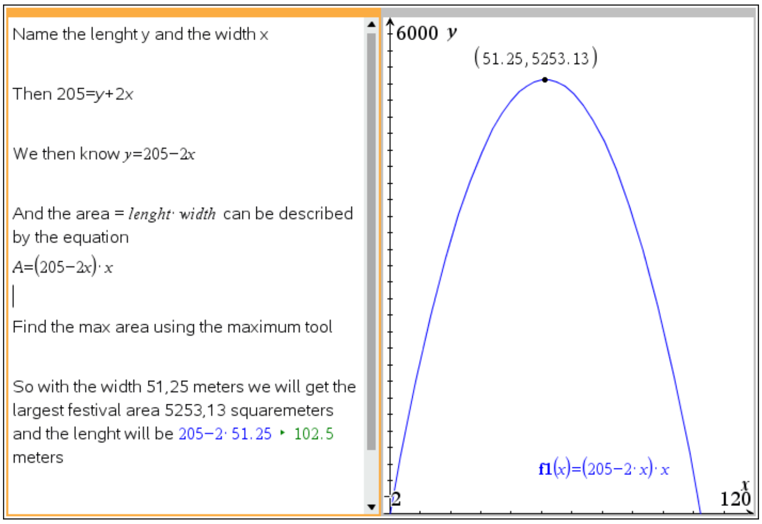
- As before, but doing the quadratic regression on the point plot to model the data.

	A lenght	B width	C area
=			=lenght*width
1	50.	77.5	3875.
2	80.	62.5	5000.
3	100.	52.5	5250.
4	110.	47.5	5225.
5	120.	42.5	5100.
6	105.	50.	5250.
7	90.	57.5	5175.
8	115.	45.	5175.
9	75.	65.	4875.
10			
11			

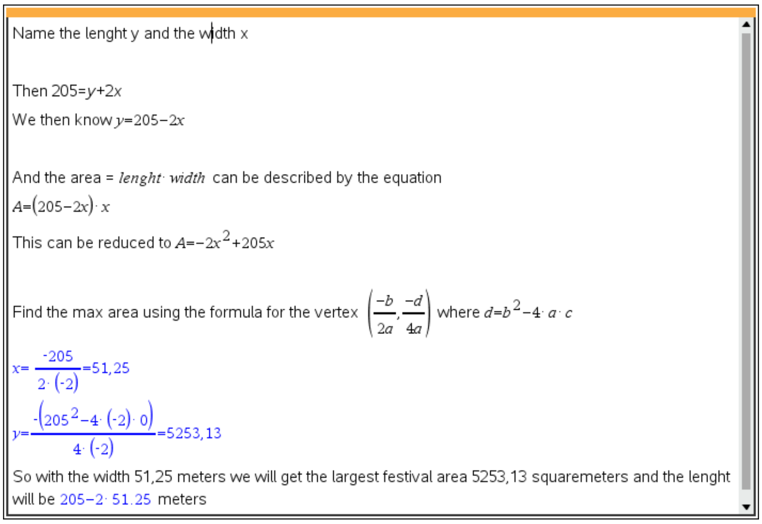
B10



- Some students might denote the length and the width by variables and write the algebraic formula for the area. They plot this expression as a graph in Ti-Nspire and determine the maximum of the function graphically.



- As in the previous remark but calculating the value of x for the maximum using the vertex formula for the resulting 2nd-degree polynomial.


Name the lenght y and the width x

Then 205=y+2x

We then know $y=205-2x$

And the area = $lenght \cdot width$ can be described by the equation

$A=(205-2x)\cdot x$

This can be reduced to $A=-2x^2+205x$

Find the max area using the formula for the vertex $\left(\frac{-b}{2a},\frac{-d}{4a}\right)$ where $d=b^2-4\cdot a\cdot c$

$x=\frac{-205}{2\cdot(-2)}=51,25$

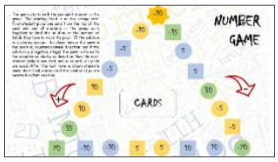
$y=\frac{-(205^2-4\cdot(-2)\cdot 0)}{4\cdot(-2)}=5253,13$

So with the width 51,25 meters we will get the largest festival area 5253,13 squaremeters and the lenght will be 205−2· 51.25 meters

The number game

Why the square root is not a negative number

Team: XV. gymnasium - 1, Zagreb, Croatia

Target knowledge	The distinction between the square root of a non-negative real number and the solution of a quadratic equation $x^2 = a, a \geq 0$.
Broader goals	Square root function, organizing data, mathematical communication, mathematical conventions.
Prerequisite mathematical knowledge	Basic mathematical knowledge in algebra (e.g., square of a number, solution of an equation).
Grade	Age 15-16 (2nd grade in Croatia)
Time	This scenario has two parts. In the first part (60 minutes) the students discover a mathematical convention by playing the game and in the second part (30 minutes) they are asked to formulate the definition on their own. One can also use only the first part of the scenario.
Required material	Game board, cards and figures (one per group), A3 posters, enlarged cards, markers, magnets for presenting definitions, clean A4 papers for the definitions, a larger game board printout for the presentation of the first round, poster with all the cards for the second validation phase, papers or notebooks to solve tasks during the game.
Problem: You are presented with the "Number Game" including a game board, playing pawns, and 12 cards. The rules of the game are written on the game board. Play the game once and explain which fields the game can end in. Define what is the winning position and explain how one can win the game and why.	

Phase	Teacher's actions, including instructions	Students' actions and reactions
Devolution (didactical) 3 minutes	The teacher divides students into groups of four, explains the rules and the organization of the "Number Game", and distributes game materials (boards, cards, and figurines). (S)he instructs students to play the game once and remember the last field that the pawn landed on.	Students listen and follow the teacher's instructions. They ask for clarifications about the rules of the game if needed.
Action (adidactical) 15 minutes	The teacher monitors how the groups play the game. If students are undecided about answering ambiguous cards, the teacher does not help and instructs students to decide on their own. The teacher attaches the larger board printout to the wall. (whiteboard).	Students play one round of the game on the board and record the field on which they finished the game.
Formulation and Devolution (didactical) 5 minutes	On the prepared game board, the teacher records the final fields for each group. If all groups got the same value, the teacher asks if it is possible to get some other values. The teacher asks:	Students formulate that there are ambiguous cards so that groups can make different choices during the game and end up in different fields.

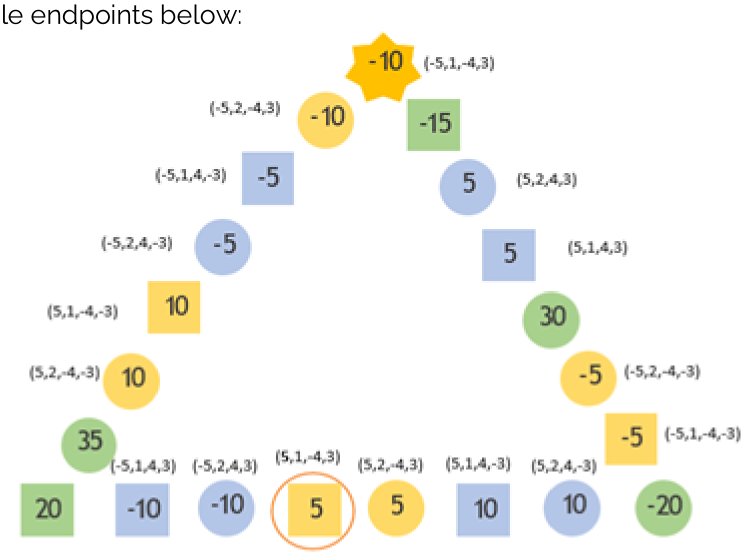
	"Why have you finished on different fields?" Explain in detail which fields the game can end in and why.	
Action (adidactical) 15 minutes	The teacher observes the work of the groups and decides on the order of presentations.	The groups analyse the steps and different possibilities of ending the game. They conclude which group of cards gives fixed shifts and which group of cards makes variable outcomes and determines possible outcomes. They prepare a presentation of the conclusions.
Formulation (didactical) 7 minutes	The teacher invites groups for a presentation starting with the group that has the least general conclusion.	The groups present the conclusions: Why are the final fields different? Which cards led to the different fields? Which fields can be reached? See "*Possible ways for students to realize the target knowledge*".
Validation (didactical) 10 minutes	The teacher puts a poster with all cards enlarged on the board and invites representatives of groups to simultaneously place different colour labels on ambiguous cards. The teacher leads a discussion about cards where there is a disagreement. The teacher draws particular attention to the cards with the square root and the card with the quadratic equation.	Students discuss and decide which cards are ambiguous and which are not. If there is a disagreement about the card with a square root, groups can compare the number of points they received at the end of the game based on the sign of the square root. The group that thinks the card with the second root is ambiguous scores a negative number of points. Students may wonder why the game was made that way.
Institutionalisation (didactical) 5 minutes	The teacher focuses on the square root. (S)he presents the historical and practical reasons for the uniqueness of the second root. The difference between the square root and the solution of the quadratic equation is emphasized. If the lesson ends in this phase, the teacher writes on the board the definition of the square root and the solutions to the quadratic equation $x^2 = a$**.** Otherwise, the lesson continues with a new devolution.	Students listen and make notes.
Devolution (didactical) 5 minutes	The teacher shows on a concrete example the difference between the square root and the solution of a quadratic equation. The teacher instructs students to write down on paper as a group the definition of the square root and the solution of the quadratic equation $x^2 = a$.	Students listen and copy the example.

Action (adidactical) 5 minutes	The teacher observes the work of the groups.	In the group, students write down on a clean A4 paper the definition of the square root and the solution(s) of the quadratic equation.
Formulation (adidactical) 7 minutes	The teacher asks the students to exchange their papers with definitions and validate the definition they have received. Each group has a number and the group with the number "n" validates the group with the number "n+1". The teacher prepares the whiteboard and invites all the groups to attach their definitions on the board at the same time.	Students work in groups to "evaluate" others' definitions, correct them, or conclude that it is correct based on what they have written as their definition. Validation is in the form of rewriting other groups' definitions and offering a counterexample when applicable. Students put revised papers on the board.
Validation (didactical) 10 minutes	The teacher leads the discussion of which definition (if any) should be accepted as an official one, and why. The teacher can recall the historical approach to the square root to guide the discussion into the conventional definition.	Students actively participate in the discussion.
Institutionalisation (didactical) 3 minutes	Based on the students' work and discussion, the teacher writes on the board the definition of the square root and the solutions to the quadratic equation $x^2 = a$.	Students copy the final definition in their notebooks.

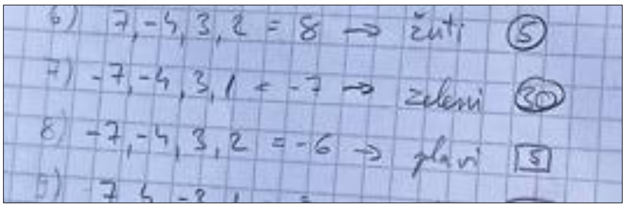
Possible ways for students to realize target knowledge

We first present the conclusions about the cards in the game. Some cards have a unique solution. The sum of unique solutions is 10. There are 3 cards with two mathematically correct solutions (absolute value ± 4, divisor of 2 (1,2) and solutions of a quadratic equation ± 3). There is a false double card: the square root of 25. The only solution is 5.

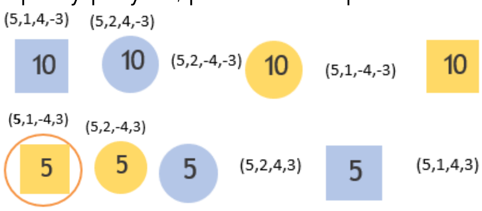
You can see all possible endpoints below:



Students may mistakenly think that "the difference of the squares of numbers 4 and 3" is ambiguous with the values ± 7. In this case, they will probably end up on fields that cannot be achieved (green fields on the board) if the game is otherwise properly played.



If the game is properly played, possible endpoints are:



The situation in which some students will reach a field with a negative number at the end of the game, while others will reach a positive number, should provoke a discussion about the differences and reasons for such different endings. When properly playing the game only choices with negative values of a square root will yield negative endpoints, hence motivating students to question the correctness of their choice.

Students might question how to deal with ambiguous cards and should be advised to make a mathematical decision in the group. If the teacher notices that the students struggle with mathematics in a task that is not relevant to the target knowledge of the square root, the teacher should assist them in reaching the mathematical answer.

It is expected that some groups will realize very early by replaying the game that the unique cards make a fixed shift from the starting position and in every subsequent game attempt that group can replay only the ambiguous cards. Those groups will finish their observations much earlier than other groups. On the other hand, if a group starts to replay the whole game using all the cards it will consume too much time and students will be able to only partially participate in the discussion of all possible endings.

Revising students' definitions of a square root the teacher might see language and notation not common in the definition, e.g., the absolute value or ignoring the non-negativity of a radicand. All of this should be validated as is.

Remark
Our advice is not to change the ambiguous cards and the false double card. For all the cards with unique answers, the teacher can change the tasks and their solutions as long as the sum remains 10.

To conclude the discussion of why a square root is defined nonnegatively the teacher may use historical background: the concept of the square root is much older than negative numbers. Indeed, we may find that the square root appeared in Babylon, 1800-1600 BCE, while the negative numbers appeared in China around 200 BCE and in India in 600 CE.

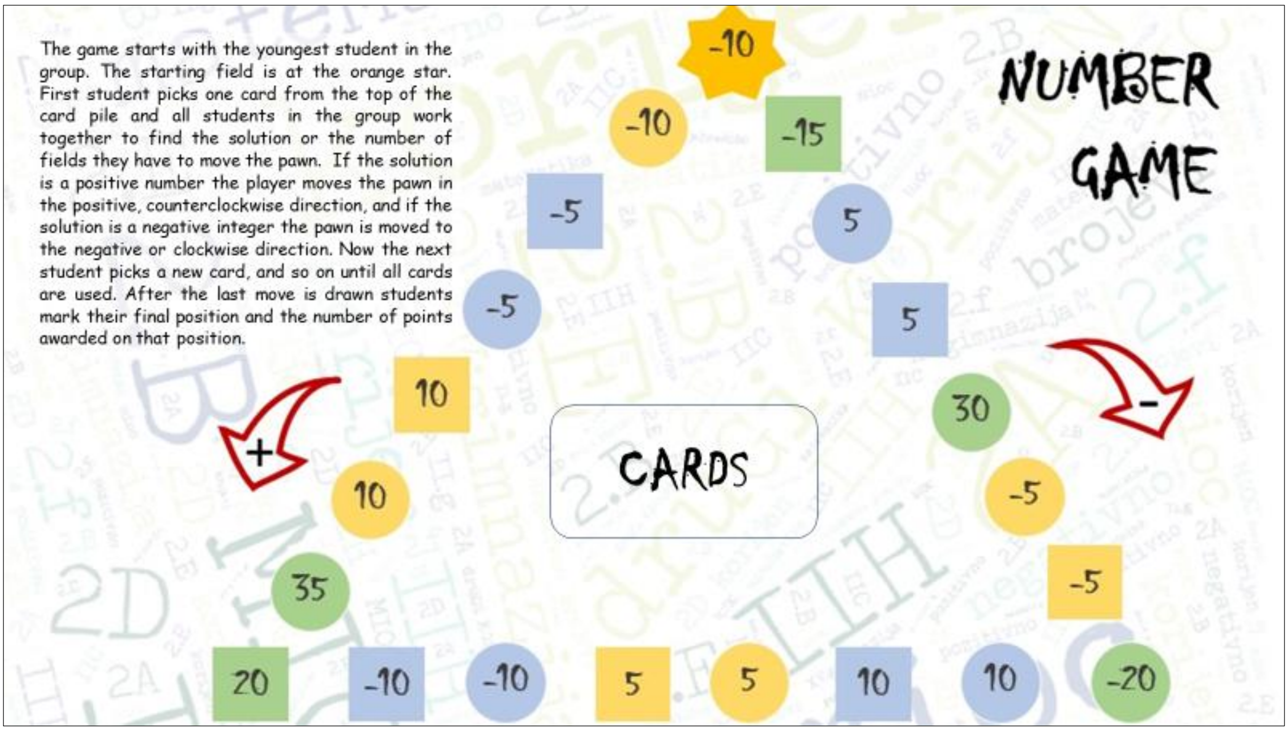
An alternative way to realize the second part of the lesson

The second part of the lesson can also be carried out in a more closed way by offering students cards with definitions in words and symbols. Students connect the appropriate cards, evaluate and, if necessary, correct the definitions.

Words	Symbols
The square root of a number a is an absolute value of a number that, when squared, equals the number under the square root.	$\sqrt{a} = \lvert x \rvert$ if $x^2 = a$
The square root is a positive number that, when squared, results in the number under the root.	$\sqrt{a} = x$, $x > 0$ and $x^2 = a$
The square root of a non-negative number is a non-negative number that, when multiplied by itself, results in that number.	For $a \geq 0$, $\sqrt{a} = x$ if $x \geq 0$ and $x \cdot x = a$
The square root of a non-negative real number a is a non-negative real number $\sqrt{a}$ that, when squared, equals a.	$\forall a \in \mathbb{R}, a \geq 0, \sqrt{a} \geq 0, \left(\sqrt{a}\right)^2 = a$
The square root of a positive real number a is a positive real number $\sqrt{a}$ that, when squared, equals a. The following statement is also valid: $\sqrt{0} = 0$	$\forall a \in \mathbb{R}, a > 0, \sqrt{a} > 0, \left(\sqrt{a}\right)^2 = a$ and $\sqrt{0} = 0$

NOTE: See the project's webpage for the following editable materials in Word.

GAME BOARD:



CARDS:

A. The sum of all solutions $(x+7)(x-6)(x+5)=0$	**B.** A positive divisor of 2	**C.** The smallest of three consecutive integers whose sum is -15	**D.** The square root of 25
E. An integer solution of the equation $3x(2x-5)=0$	**F.** The difference of the squares of the numbers 4 and 3	**G.** The cube of the sum of the numbers 1 and 3 divided by 8	**H.** A number whose absolute value is 4
I. A number whose square is 9	**J.** The arithmetic mean of the Pythagorean triple $(5, 12, ?)$	**K.** A number opposite to the number of diagonals of a convex pentagon	**L.** The median of the given numbers: $-\frac{7}{2}, -11.5, 0, -\sqrt{3}, -3$

Fruit for snack

Domain restriction needed for the inverse function

Team: XV. gymnasium - 1, Zagreb, Croatia

Target knowledge	Domain restriction as a precondition for the existence of an inverse function.
Broader goals	The square root function as the inverse of the quadratic function. Establishing requirements to construct an inverse function. Making a given function injective and surjective. Mathematical communication.
Prerequisite mathematical knowledge	The notion of a function.
Grade	Age 16 (2nd grade in Croatia)
Time	80 minutes
Required material	Handouts with the formulation of the given problem, diagrams, and graphs.

Problem:
Today we will deal with the problem of fruit distribution under the given conditions. Examine the given data and discuss possible ways of fruit distribution on each day. Draw the distributions in the given coordinate systems.

Phase	Teacher's actions, including instructions	Students' actions and reactions
Devolution (didactical) 10 minutes	The teacher divides students into groups, explains the problem, and shares the handouts with the wishes, and diagrams. (S)he assigns the 1st task.	Students listen and study the set up.
Action (adidactical) 15 minutes	The teacher observes the students' work without interference.	Students study the diagrams and connect them to the days based on the description. For each day they notice whether the girls broke some rules and whether the distribution is possible.
Formulation (didactical/ adidactical) 10 minutes	The teacher leads the discussion about the students' findings. It is also expected that the students will decide which girl to give fruit to on days when two or more want the same fruit.	The groups present conclusions: when distributions are possible, when they are not and how to distribute fruits.
Devolution (didactical) 5 minutes	The teacher shares the coordinate systems and assigns the second task.	Students listen and consider the questions on the handout.
Action (adidactical) 5 minutes	The teacher observes the student's work without interference.	Students draw the distributions in the coordinate systems.
Formulation (adidactical) 10 minutes	The teacher invites the students to share their answers on the blackboard or using an online tool.	Students present their solutions. The mothers present after the others.
Validation (didactical/ adidactical) 15 minutes	The teacher leads the discussion in which students compare their results.	The groups present and compare graphs. Students notice that on days when more girls want the same fruit, distributing can be done in different ways and they are all valid.

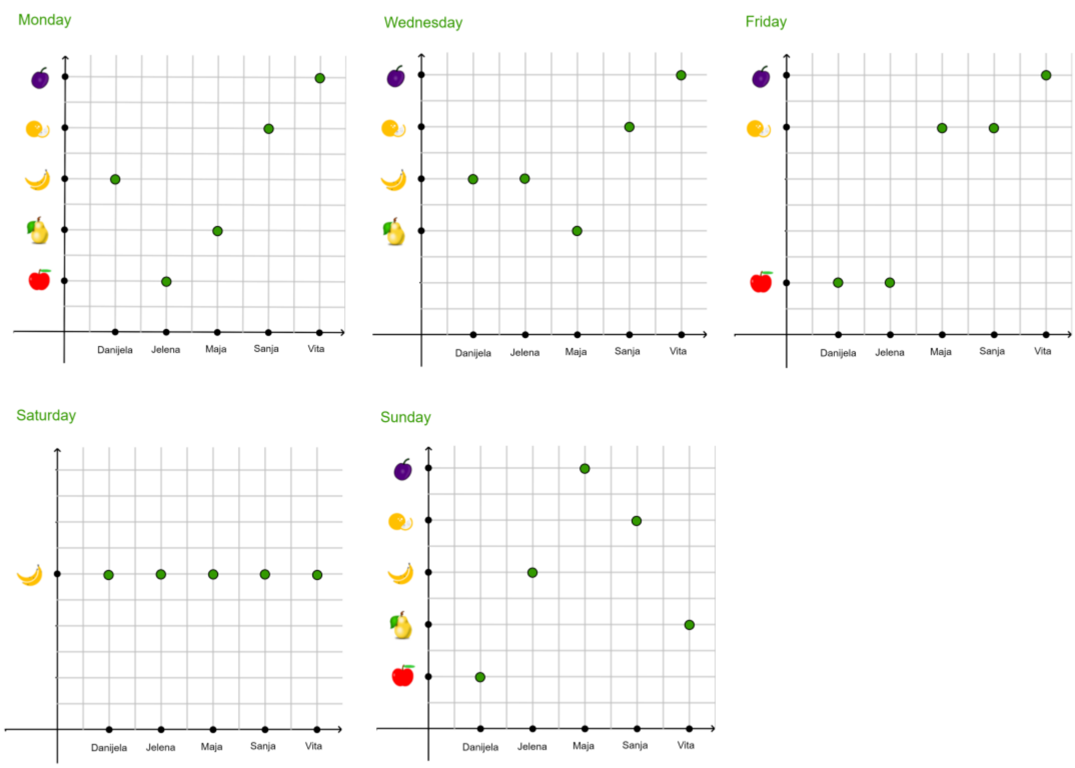
Institutionalisation (didactical) Time estimate 10 minutes	The teacher asks the students to connect the situations and objects from fruit distribution with mathematical concepts and introduces the terms *inverse function* and *domain restriction*.	Students connect situations with the mathematical concepts: domain, codomain, and the assigning rule. They notice two different mappings of "wishing" and "distributing" and their inverse relationship.

Possible ways for students to realize target knowledge

Students should first notice that on Tuesday and Thursday the girls did not follow the rules. On Tuesday one girl did not wish for any fruit, while on Thursday one girl wished for two fruits. Hence, the data for these days are relations that do not represent functions and the diagrams are not given for these days.

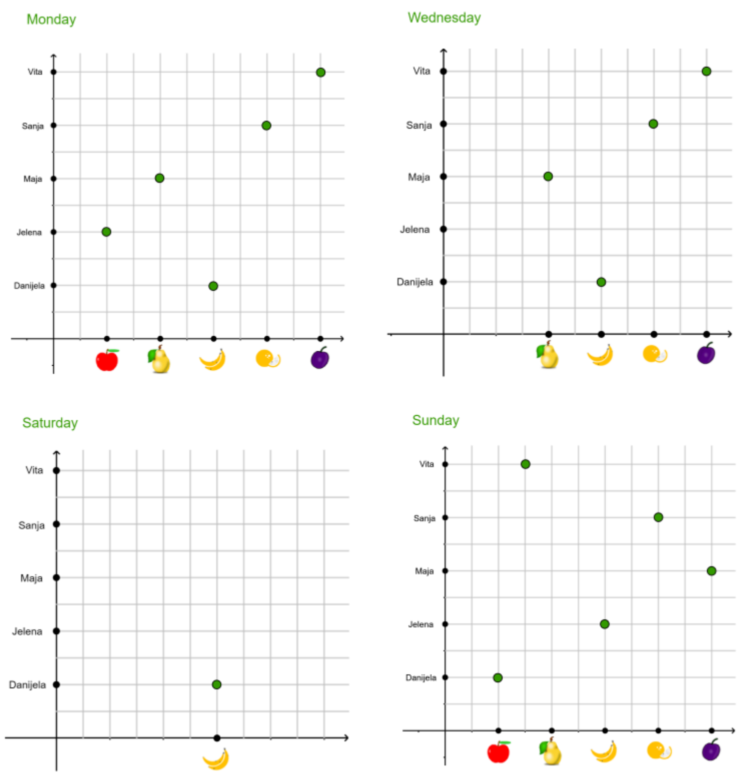
For the other days, the students will probably find many different strategies to associate the descriptions of the wishes with the diagrams. For example, they can consider the wishes of the first two girls to see which diagram corresponds to which day.

Drawing the wishes in the (girl, fruit)-systems is a straightforward task if the students have encountered it before. Some students might be confused because the graph does not represent numbers or because it is a discrete graph. The solution for these graphs is given in the following picture.



The "mothers" have the (fruit, girl)-systems. For these students, the task requires discussion before making any decisions. The students might discuss different aspects from everyday life, e.g., whether it is fair that some girls will not receive a fruit, whether it is allowed to cut the fruit in more pieces, whether it is important to make the distribution fair when considering all the days (e.g., if one girl did not get fruit on one day, will she get it on the next day) etc.

Distributions are uniquely determined on Monday and Sunday. On Wednesday, Friday, and Saturday they should decide how to distribute the fruit. One possible choice is presented in the following picture.



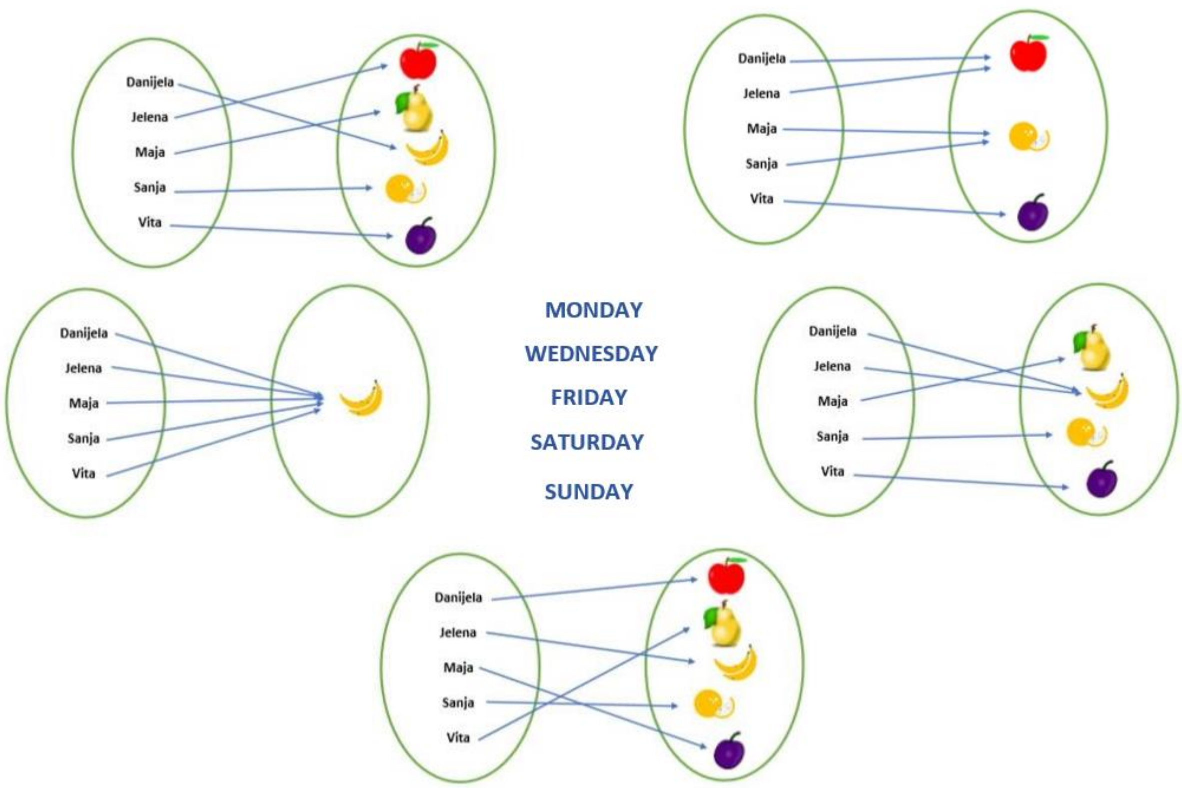
The main part of the lesson is the discussion in which the students gradually use the mathematical language: functions, domain, restriction, inverse, etc. It is expected that from the beginning of the lesson the students will use informal descriptions related to the concrete given context. Some phrases and formulations will remain during the lesson and will be rephrased only by the teacher, but we also expect that when introduced to the coordinate systems the language will change and more mathematical terms will be introduced in the descriptions by the students themselves.

NOTE: See the project's webpage for the following editable materials in Word.

HANDOUT

Five sisters (Danijela, Jelena, Maja, Sanja, and Vita) are given fruits for a snack. They agreed with their mother that each of them would choose one fruit each day. There is a fruit basket on the table, in which there is one fruit of each kind every day.
The mother asks who wanted what fruit and gets answers for each day.

- Monday: Danijela (banana), Jelena (apple), Maja (pear), Sanja (orange), Vita (plum)
- Tuesday: Danijela (no fruit), Jelena (apple), Maja (pear), Sanja (orange), Vita (plum)
- Wednesday: Danijela (banana), Jelena (banana), Maja (pear), Sanja (orange), Vita (plum)
- Thursday: Danijela (banana), Jelena (apple), Maja (pear), Sanja (orange), Vita (plum and pear)
- Friday: Danijela (apple), Jelena (apple), Maja (orange), Sanja (orange), Vita (plum)
- Saturday: Danijela (banana), Jelena (banana), Maja (banana), Sanja (banana), Vita (banana)
- Sunday: Danijela (apple), Jelena (banana), Maja (plum), Sanja (orange), Vita (pear)



TASK 1
Your task is to determine which diagram represents which day and to suggest a way of fruit distribution for each day. Did the girls follow the rules every day? Discuss whether every girl can get the desired fruit or not. Why not? Who can and who can't get the fruit?

TASK 2
Show the distribution in the coordinate system. Some groups have coordinate systems (girls, fruits) for the wishes. Others (the "mothers") have coordinate systems with reversed coordinate axes and they decide how to distribute the fruit on days when two or more girls ask for the same fruit.

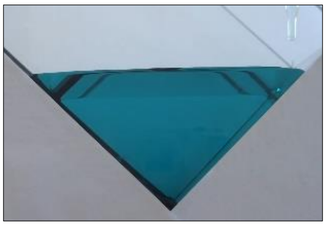
Volumetric flow rate

Modelling a real-life situation with the square root function

Team: XV. gymnasium - 1, Zagreb, Croatia

Target knowledge	Discovering the square root dependency in a real-life situation.
Broader goals	The relationship between the quadratic function and the square root function. Domain restriction. Calculation and manipulation with various measuring units. Expressing a dependency rule as a formula. Sketching and scatter plotting the graph of the square root function on paper or using ICT tools. Inquiry skills: data organization and analysis, interpretation of the results (argumentation) during the presentation. Interdisciplinary skills: connection of the notations and procedures between the mathematics and physics approaches.
Prerequisite mathematical knowledge	Basic knowledge of functions, basic understanding of the constant rate of change (constant speed)
Grade	Age 16 (2nd grade in Croatia)
Time	90 minutes, two school periods
Required material	Computer/tablet, calculator, ruler. Large papers for presentation (self-adhesive paper). Video clips: https://bit.ly/watervideopart1, https://bit.ly/watervideopart2

Problem:
The video clip shows the filling of the prism with water at a constant rate. Watch the video at your own pace and describe how the height of the water in the prism depends on the filling time. Estimate the height of the water after 5 minutes, 5.5 minutes and 6 minutes based on your model.
Justify your answer to the best of your ability using mathematical arguments (graphs, tables, formulae, etc.)



Phase	Teacher's actions, including instructions	Students' actions and reactions
Devolution (didactical) 5 minutes	The teacher divides students into working pairs, shares the first part of the video https://bit.ly/watervideopart1 and the assignment to describe the dependency of the water height on the elapsed time.	Students listen and access the video through the shared link.
Action (adidactical) 35 minutes	The teacher walks through the classroom, observing students' activities without interfering.	Based on the video students describe how the water height depends on the elapsed time. They might decide to measure, to replay the video multiple times and to record data in different formats.
Formulation (didactical/ adidactical) 10 minutes	The teacher distributes the presenting papers/posters and invites students to prepare their conclusions for the presentation. The teacher follows the students' work and decides on the presenting order starting with the pair with the least general conclusions.	Students write down their conclusions on the presenting papers. The expected answers are discussed below, see "*Possible ways for students to realize the target knowledge*".

Devolution (didactical) 5 minutes	The teacher asks the students to estimate the height of the water after 5 minutes, 5.5 minutes and 6 minutes based on their model. They should write their estimates on paper. The teacher shares the link for the complete video https://bit.ly/watervideopart2	Students listen and prepare to write down the answer. They access the complete video through the shared link.
Action (adidactical) 10 minutes	The teacher walks around, observing how the groups are answering the questions. If needed, the teacher instructs students to check how accurate their estimates were using a video: "Write your measured heights on the paper next to your estimated heights. What does this imply regarding your model and or assumptions?"	Students answer the questions and write their answers on the presenting paper.
Formulation (adidactical) 5 minutes	The teacher invites students to present their posters.	Pairs place their presenting papers (self-adhesive papers) on the board. Afterwards, each pair, one by one, presents their work and conclusions.
Validation (didactical/ adidactical) 10 minutes	The teacher organizes a discussion in which students compare their working strategies and results.	Students follow their peers' work and ask additional questions. They discuss different strategies and their conclusions. Students verify the accuracy of their results and discuss the validation process.
Institutionalisation (didactical) 10 minutes	Based on the students' work and presentations, the teacher states the conclusion: water height in the prism that is filling with a steady uniform flow depends on the time following the square root function. The teacher states the definition of the square root function stating the domain, the codomain, and the rule.	Students listen, converse with the teacher, and actively participate in the definition construction. They write down the definition and the activity conclusions.

Possible ways for students to realize target knowledge

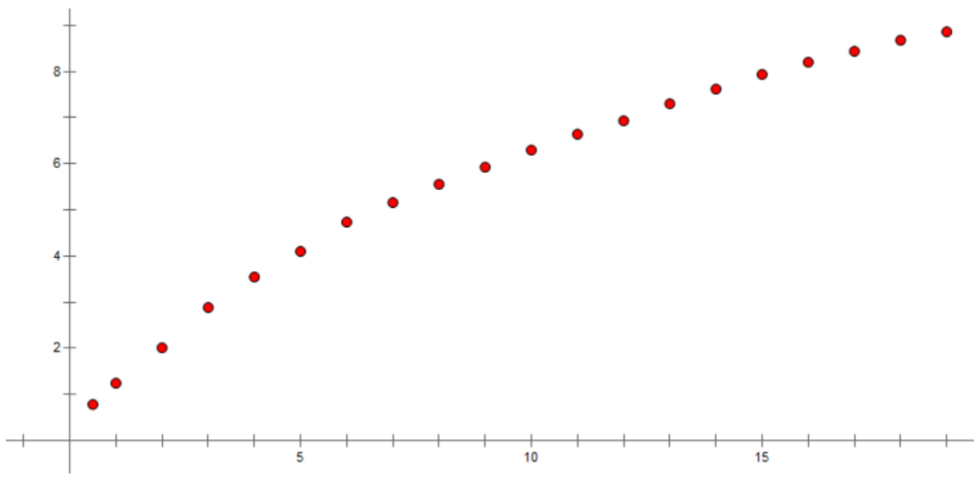
In the first part of the lesson, we can expect the students to conclude that the height-time dependency is an increasing relation as with more time elapsed the height is larger. Probably they will notice that the dependency is not linear, or even more precisely that at the beginning the height increases at a higher rate, and then the height change slows down.

Students might use different measuring units for height (mm, cm, m, and pixels) and time (seconds, minutes). They might conclude that the height is not proportional to the time since in the doubled time the height does not double (after 20 seconds height is 3.5 cm while after 40 seconds the height is 6.1 cm).

Students might assume that the relationship is a square root function and might attempt to find the function rule based on the plotted points. Also, by plotting points in the coordinate plane, they might notice that the trend is not linear (points are not colinear).

Students might recognize the fit of the dependency using graphic methods; by stopping the video and measuring the height on the screen in the given moment and plotting time and height in the coordinate plane. The following is an example of students' recorded measurements and corresponding plot:

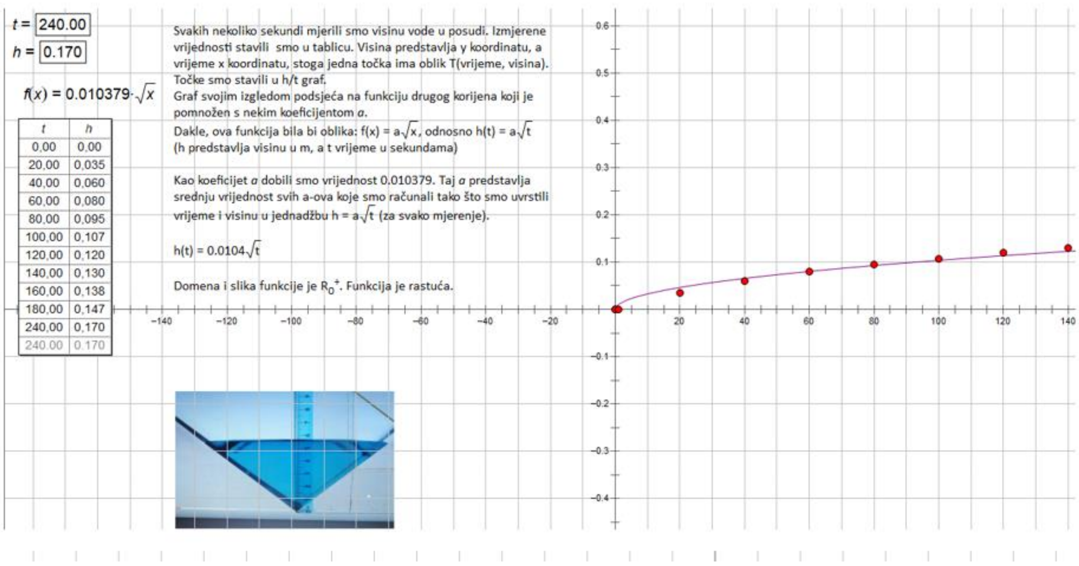
t	h
0,5	0,78
1,0	1,24
2,0	2,01
3,0	2,88
4,0	3,55
5,0	4,10
6,0	4,74
7,0	5,16
8,0	5,56
9,0	5,93
10,0	6,30
11,0	6,64
12,0	6,93
13,0	7,30
14,0	7,62



Considering the shape of the graph above students might notice that the height-time dependence is a square root function while the function $f(x) = \sqrt{x}$ does not accurately fit the data (e.g., $\sqrt{20} \neq 3.5$).

If they assume that the relation is generalized as $f(x) = a\sqrt{x}$, the value of constant a can be obtained from measured values and by drawing the graph of the function f they can check if the plotted points belong to the graph. Some errors might occur due to measuring errors.

Some students will find the constant a for each measurement and then average all those values. Others might vary the parameter a to fit its value to the plotted points.

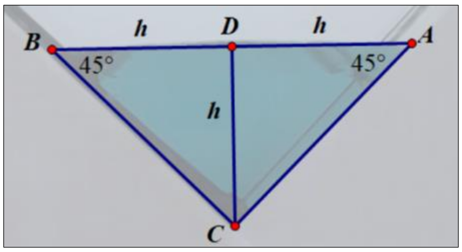


Students can validate their models (estimations) by drawing a graph of the derived function and by checking if their measured points belong to the function graph. Based on the complete graph they can check if the derived function accurately models the height vs time dependency throughout the whole video.

Students might also derive the dependency rule using geometry and the physical concept of flow rate.

Geometrically, students can find the area of an isosceles right triangle observed in the video $\frac{1}{2} \cdot h \cdot 2h = h^2$.

This triangle is the cross-section (base) of the prism container. If the width of the container is d, then the volume of the container is $V = h^2 d$.



Let v be the flow rate speed, i.e., the speed with which the water is flowing from the pipe.
If A is the cross-sectional area of the pipe, then the volumetric flow rate is $Q = Av$.
The volumetric flow rate is the change of volume over a unit of time, $Q = \frac{\Delta V}{\Delta t}$.
Since the initial values are $V_0 = 0$, $t_0 = 0$, we can say that $V = Qt = Avt$.

As A and v are constant, the only important conclusion from physics can be stated in a simple form: the volume V depends linearly on time and the constant of proportionality is the flow rate Q.

By equating the two expressions for the volume, it follows:

$$h^2 d = V = Qt.$$

So, we have

$$h^2 = \frac{Q}{d} \cdot t, \qquad \text{i.e.} \qquad h = \sqrt{\frac{Q}{d}} \cdot \sqrt{t} = a\sqrt{t}.$$

Some students might get confused by the fact that the width of the container d is not known and they might think that the problem cannot be solved because of that. It is enough to explain that the container is a prism, so its width is not important, and they can consider the problem only in terms of area instead of volume.

The considerations about the volume are more realistic as one might, in general, measure the size of the pipe and the speed of the water, but this is irrelevant as the students only have the video and no access to the actual pipe. This is seen in the above formulas as Q and d are jointly replaced by a single parameter a.

Some students might intuitively understand that the width is not important, and they will consider the flow rate only for the area. Regardless of the level of justification and awareness that the students have about this simplification, their formulas might be as above only with $d = 1$.

Path towards the fence

Perpendicular lines

Team: XV. gymnasium - 1, Zagreb, Croatia

Target knowledge	The product of slopes of mutually perpendicular lines, not parallel to coordinate axes, equals -1.
Broader goals	Applying algebra and geometry in modelling and problem-solving situations. A mathematical justification.
Prerequisite mathematical knowledge	Linear function. Similarity.
Grade	Age 15 (1st grade in Croatia)
Time	60 minutes
Required material	Worksheet including task and empty coordinate system, set squares and rulers.

Problem:
You are invited to collaborate with the developing team for computer games. The Player's position is given, and he is to move along the shortest path towards the fence, both drawn in the diagram.

a) Your goal is to derive the formula that describes the player's movement.
b) During the game, the fence randomly rotates about the lower left corner. For a given general position of the fence find the corresponding player's route.

Use mathematical terminology and arguments to validate your solution.

Phase	Teacher's actions, including instructions	Students' actions and reactions
Devolution (didactical) 5 minutes	The teacher assigns students to the working groups and presents the problem.	Students listen.
Action (adidactical) 20 minutes	The teacher observes the students' work.	Students conclude that the player must move along a straight line perpendicular to the fence. They draw the line through the player's position orthogonal to the fence and derive its equation. In this explicit example, they might compare the slopes of both lines. Students solve the problem in general position: they might use the similarity of triangles, Pythagoras's theorem, properties of line gradients, and trigonometry of right-angled triangles.
Formulation (didactical/ adidactical) 10 minutes	The teacher hands out self-adhesive whiteboard sheets and instructs them to write down their conclusions and arguments for the presentation.	Students copy their results, conclusions and arguments and prepare to present them.
Validation (didactical/ adidactical) 15 minutes	The teacher invites the teams to present, starting with the team that has the least general results and/or conclusions and arguments.	Students present their work and follow other teams' presentations. Students verify the accuracy of the results.
Institutionalisation (didactical) 10 minutes	The teacher writes down on the whiteboard the property of slopes for mutually perpendicular lines.	Students listen, converse with the teacher, and actively participate.

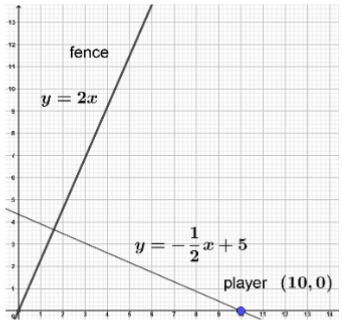
	The teacher proves the stated property and if needed completes the students' justification that was presented. The teacher initiates the discussion of whether the property holds for all lines and point positions.	

Possible ways for students to realize target knowledge

a) Students conclude that they need to derive the equation of the line perpendicular to the given line of the fence. It is not crucial for students to find the equation of the fence; they can just draw the perpendicular line and find its equation.

b) Students also might:

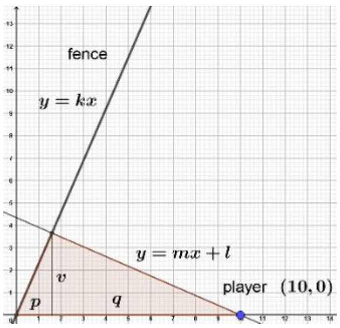
Experiment with various but specific fence positions and base their conclusions on them that the product of slopes of mutually perpendicular lines equals -1. In that case, justification is not complete.



Conclude that signs of the slopes must be opposite since if one is increasing the other one must present a decreasing function. Since in the given example, the fence has a positive slope the player's route must have a negative slope.

Conclude that if the slope of the desired line is m and the line passes through the point $(10,0)$ its equation is $y = mx - 10m$.

- Apply the similarity of triangles (or Euclid's theorem)

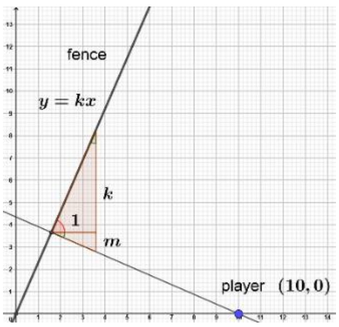


$$v : p = q : v$$

$$k = \frac{v}{p}, m = -\frac{v}{q}$$

$$\Rightarrow m = -\frac{1}{k}$$

- Draw triangles like in the diagram. The slope of the fence is k so when x increases by 1, y increases by k. Triangles are similar due to the equivalent angles. Hence,

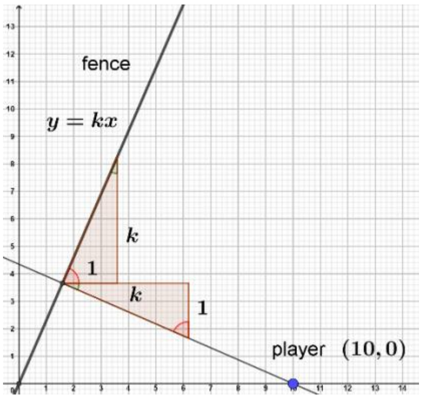


$$1 : k = m : 1$$

$$m = \frac{1}{k}$$

The slope of the desired line is $-m$ as x increases by 1, y decreases by m.

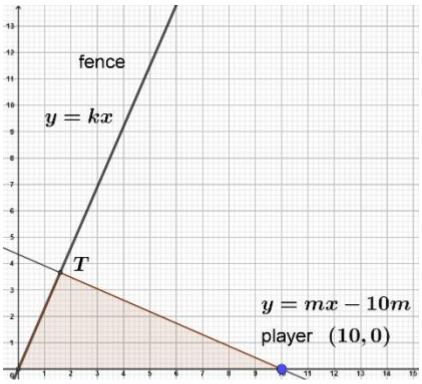
- Draw similar triangles with sides 1 and k. Triangles might be expanded since the sum of angles in a right-angled triangle is $90°$.



The slope of the perpendicular line is

$$m = \frac{\Delta y}{\Delta x} = -\frac{1}{k}$$

- Write down the equation of the perpendicular line through the point $(10,0)$, find the point of intersection and apply Pythagoras's theorem.



Coordinates of the point $\boldsymbol{T}$:

$$y = kx$$

$$y = mx - 10m$$

$$kx = mx - 10m \Longrightarrow x = \frac{10m}{m-k}$$

$$y = kx = \frac{10mk}{m-k}$$

Applying Pythagoras's theorem

$$10^2 = \left(\frac{10m}{m-k}\right)^2 + \left(\frac{10mk}{m-k}\right)^2 + \left(\frac{10mk}{m-k}\right)^2 + \left(10 - \frac{10m}{m-k}\right)^2$$

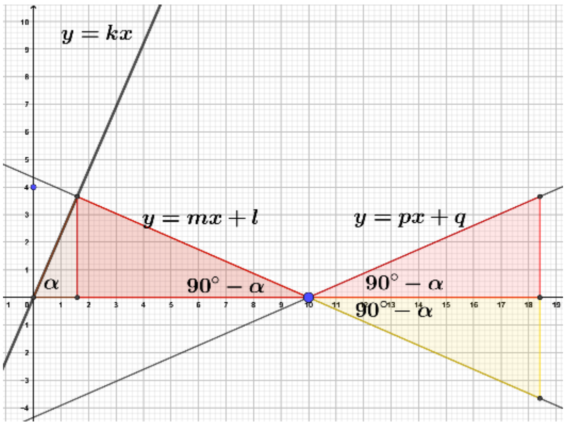
Simplifying

$$100(m-k)^2 = 100m^2 + 200m^2k^2 + 100k^2$$

$$m^2 - 2mk + k^2 = m^2 + 2m^2k^2 + k^2$$

$$-1 = mk$$

- Apply trigonometry of a right triangle

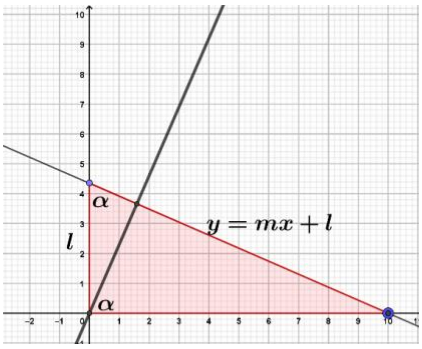


$$k = \operatorname{tg}\alpha = \frac{\sin\alpha}{\cos\alpha}$$

$$p = \operatorname{tg}(90^0 - \alpha) = \frac{\sin(90^0 - \alpha)}{\cos(90^0 - \alpha)} = \frac{\cos\alpha}{\sin\alpha} = \frac{1}{k}$$

$$m = -p = -\frac{1}{k}$$

- Express the coefficients of the perpendicular line using the angle α



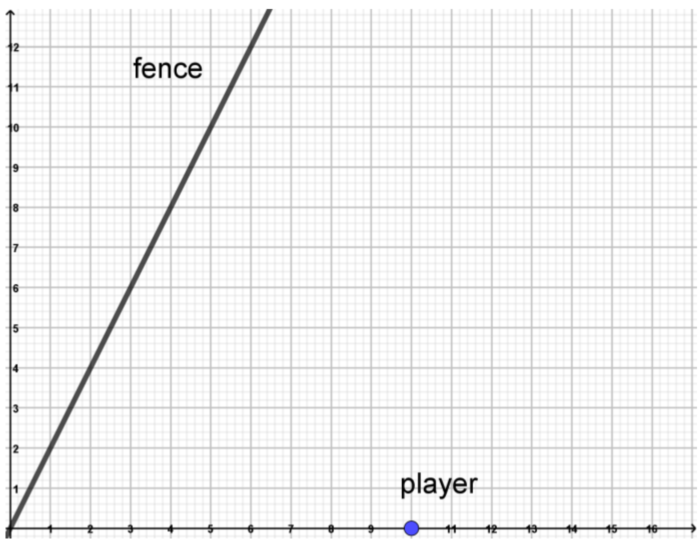
$$\operatorname{ctg}\alpha = \frac{l}{10} \Rightarrow l = 10\operatorname{ctg}\alpha$$

$$m = -\frac{l}{10} = -\operatorname{ctg}\alpha$$

$$y = -\operatorname{ctg}\alpha \cdot x + 10\operatorname{ctg}\alpha$$

NOTE: See the project's webpage for the following editable materials in Word.

HANDOUT



And chirps, chirps the cricket

Optimal linear model for the two variable data set

Team: V. gymnasium - 1, Zagreb, Croatia

Target knowledge	Argumentation of the choice of a linear model describing a discrete two-variable data set.
Broader goals	Development of mathematical language. Understanding that the same data can have various graphical representations. Interpretation and selection of an appropriate graphical representation. Real-world problem-solving.
Prerequisite mathematical knowledge	Plotting data as points in the Cartesian coordinate system in two dimensions. Linear function and its graph.
Grade	Age 16-17 (3rd grade in Croatia)
Time	60 minutes
Required material	Handouts, computers, A3 sheets of paper, markers, magnets, GeoGebra applet: https://www.geogebra.org/m/ar29c527
Problem: The table shows the results of measurements that describe the frequency with which crickets chirp at a certain temperature. To predict the frequency at an arbitrary temperature, determine the linear model that you think (best) shows the temperature dependence of the chirping frequency. Argue why you chose that particular linear model. 	

Phase	Teacher's actions, including instructions	Students' actions and reactions
Devolution (didactical) 5 minutes	The teacher divides students into groups of three and gives each group a handout with the problem. The teacher presents the problem and gives instructions. (S)he points out that the students should prepare a short presentation of their work for the rest of the class.	Students listen, follow the teacher's instructions, and ask questions if something is not clear to them. They prepare papers, a computer, and any other equipment they think will be useful.
Action (adidactical) 30 minutes	The teacher monitors the students' work and notes the strategies that different groups use. Based on this insight he prepares the order in which the groups will be invited to present their work. In case some of the students use the existing commands of one of the digital tools to get the best fit line, the teacher asks the group for their opinion on how the computer calculates the equation of that line and how they would calculate the coefficients by hand.	Students read the problem and analyse the data. Some of the students will draw their lines on paper, and some with the help of a computer. The students using computers are expected to use pre-set commands to determine the best fit line. Expected students' strategies are described at the end of the scenario.
Formulation (adidactical) 10 minutes	The teacher invites one group after another to present their linear model	One student from each group presents and argues how and why they chose their linear model. Each presentation

	and inputs their equation of the line in the GeoGebra applet. After drawing, the teacher hides the line but does not delete it. At this stage, the teacher does not comment on the students' solutions.	takes a few minutes, and other students can ask questions.
Validation (didactical) 10 minutes	In GeoGebra, the teacher shows all the lines suggested by the students and adds the line obtained by the method of least squares. (S)he invites students to discuss how the frequency predictions differ for a certain temperature (e.g., for 38 °C) and to argue which of the drawn lines they consider "the best".	Students participate in the discussion. They compare the frequency values in different models and share their thoughts on the "true value". Observing all the lines, they reflect on the meaning of "the best" linear model and give their conclusions.
Institutionalisation (didactical) 5 minutes	The teacher summarizes the arguments presented by the students during the lesson and emphasizes the different strategies the students used to choose the linear model. The teacher notes that the linear model presented in the previous phase was obtained by the method of least squares and that the basis of this method is to minimize the sum of squares of the difference between experimental and theoretical values. (S)he announces that this will be the topic of one of the next classes. The teacher asks the students for anonymous feedback: What did you learn? What questions rose up for you?	Students write down the conclusions and ask new questions that the teacher may answer in one of the next lessons. Students answer the questions on paper or via an online form.

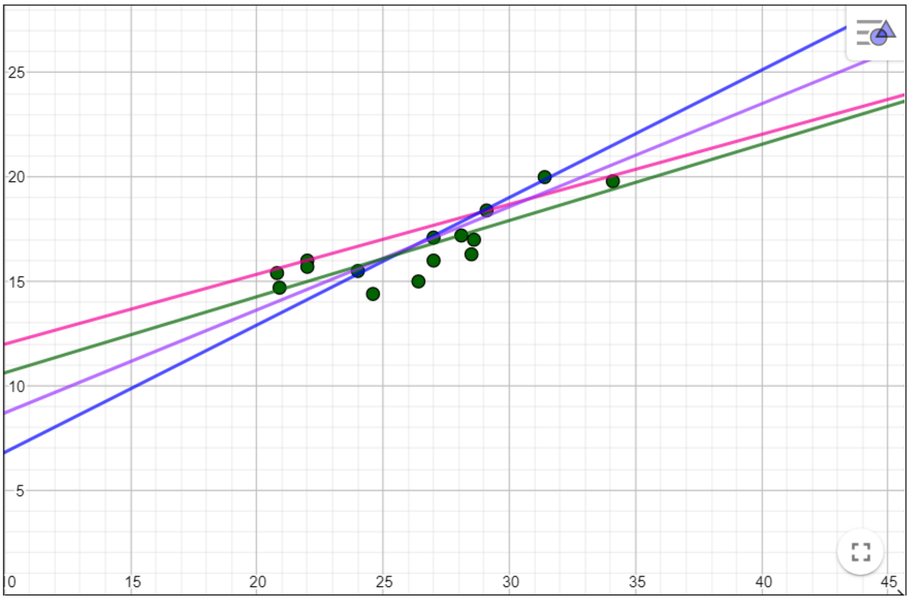
Possible ways for students to realize target knowledge

During the action phase, students might

- define the parameters for slope and y-intercept and find a line that looks the best by changing them
- determine the equation of the line through two points - various selections of these points are possible:
    - the point with the lowest temperature and the point with the highest temperature
    - a point whose coordinates are the arithmetic mean of the temperature and the arithmetic mean of the chirp frequency and one randomly selected point
    - the point whose coordinates are the arithmetic mean of the temperature and the arithmetic mean of the chirp frequency and the point whose coordinates are the arithmetic mean of the minimum and maximum temperature and the arithmetic mean of the minimum and maximum chirp frequency
    - two points that they believe best describe the given data set
    - two randomly selected points

- determine the point whose coordinates are the arithmetic mean of the temperature and the arithmetic mean of the chirp frequency and then define the slope parameter and choose the direction by changing it
- determine the equations of the lines through all possible pairs of points and then determine the slope as the arithmetic mean of all those slopes and the y-intercept as the arithmetic mean of all the obtained y-intercepts
- draw a regression line using an existing command of one of the digital tools

EXAMPLES OF STUDENTS' LINES



Note: on the abscissa is the temperature in ˚C, and on the ordinate is the chirp frequency.

EXAMPLES OF STUDENTS' ARGUMENTATION

- it passes through most points
- it is the closest to all points
- error is the smallest
- the sum of distances from points to the line is the smallest
- the sum of differences between measured and theoretical data is the smallest
- the sum of distances (absolute values) between measured and theoretical data is the smallest the computer calculated it

Remark

After this introduction comes the theory of the mathematics behind linear regression (based on the least squares method). Students are not expected to actually do these calculations by hand but rather to understand how it is done. Then they are to learn how to use a computer to calculate the best fit line and play with data to see that leaving out some of the data can change the function significantly.

This topic goes beyond the subject of high school mathematics, so we expect the students to use this knowledge in physics, chemistry, and biology.

NOTE: See the project's webpage for the following editable materials in Word.

HANDOUT

And chirps, chirps the cricket *on the black spruce knot*
Its deafening trochee, its sonorous, heavy iambic.
It's noon. - Like water, it spreads in silence
Sunny dithyramb.
(...)

Original poem by Vladimir Nazor, translated to English by Milena Ćulav Markičević

Crickets make a sound by rapidly scraping their wings over each other. Pierce (1948) mechanically measured the chirp frequency (number of wing vibrations per second) of crickets at different temperatures. Since crickets are cold-blooded, the speed of their physiological processes and their overall metabolism are affected by temperature. Therefore, there is reason to believe that temperature would have a large effect on their behaviour, such as the chirping frequency. It was generally found that crickets did not chirp at temperatures colder than 15 ˚C or warmer than 38 ˚C.

Here are the results of Pierce's measurements:

Temperature (˚C)	Chirps per second
31.4	20.0
22.0	16.0
34.1	19.8
29.1	18.4
27.0	17.1
24.0	15.5
20.9	14.7
22.0	15.7
20.8	15.4
28.5	16.3
26.4	15.0
28.1	17.2
27.0	16.0
28.6	17.0
24.6	14.4

Describe this data with a linear model.

Prepare a short oral presentation of your model in which you will tell us which linear function you chose and how you calculated it.
Explain the mathematical background of your model and argue why you consider this model to be the best linear model.

The logarithmic scale

The difference between the linear and the logarithmic scale

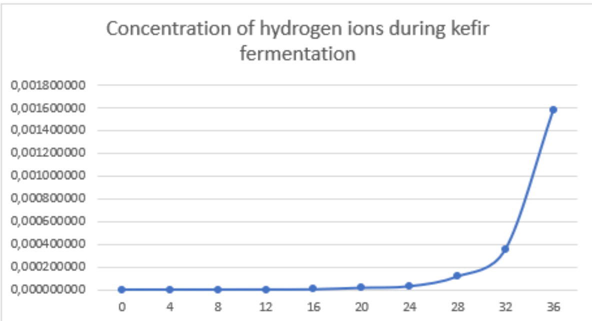
Team: V. gymnasium - 1, Zagreb, Croatia

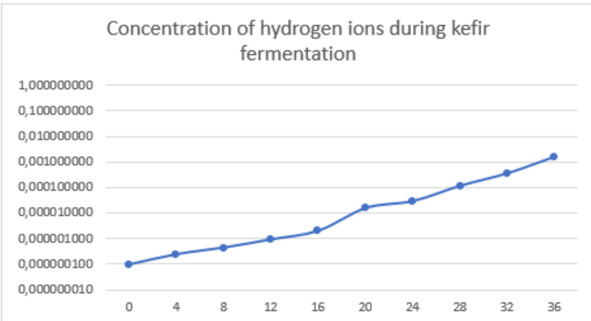
Target knowledge	Discovering and applying logarithmic scale. First activity: choosing an appropriate graphical representation and understanding various graphical representations of the same data to see the necessity of scales other than linear. Second activity: comparing and observing differences (costs and benefits) of linear versus logarithmic scale.
Broader goals	Understanding various graphical representations of the same data. Choosing appropriate graphical representation. Real-world problem-solving.
Prerequisite mathematical knowledge	Drawing and using the Cartesian coordinate system in two dimensions to plot points and read coordinates of given points.
Grade	Age 16-17 (3rd grade in Croatia)
Time	60 - 90 minutes
Required material	Handouts (four prints on one A4 paper). Blank A3 papers for drawing graphs. Markers. Big rulers. Board magnets. Worksheets printed on A4 paper. PowerPoint presentation (see project's webpage to download).

First problem:
Consider the data given in the worksheet and graph it in a coordinate system.

Second problem (example for one group):

KEFIR FERMENTATION





Answer the following three questions as a group.

1. Estimate the concentration of hydrogen ions in the 40th hour.

2. Between which two moments did the biggest change in the concentration of hydrogen ions occur?

3. Concentrations of hydrogen ions above 0.003 prevents fermentation, lumps of kefir may dissolve, and other, undesirable, bacteria may appear. After what fermentation time can kefir cause poisoning?

Then prepare a short presentation for the rest of the class in which you will explain what you concluded from the given graphs, which conclusions were easier to draw from the left, which from the right graph and which of the graphs offers you better information. You do not need to answer questions 1-3 during the presentation.

Phase	Teacher's actions, including instructions	Students' actions and reactions
Devolution (didactical) 5 minutes	The teacher divides the class into eight groups of three (ideally) and gives each group a worksheet consisting of a table with raw data and the task to represent the given data graphically in a coordinate system. Each group gets an A3 blank piece of paper, a marker, and a big ruler. There are four different tasks - two groups have the same task.	Students listen and follow the teacher's instructions.
Action (adidactical) 30 minutes	The teacher is monitoring group work.	Students are reading the task and trying to represent the given data graphically. We expect the students to follow one of the strategies described at the end of the scenario.
Formulation (adidactical) 15 minutes	The teacher instructs groups to present their work: groups with the same task hang their graphs on the board side by side and one student from each group explains in a few minutes what data they had, what problems they encountered and how they solved them. At this moment, the teacher doesn't comment on whether their representations are accurate or not.	All groups present their work following the teacher's instructions.
Devolution (didactical) 3 minutes	In the second phase, the teacher gives each group a worksheet with two graphical representations of the data they were working on (one has y-axis values on a linear and the other on a logarithmic scale) and a set of three questions based on which they need to decide which representation is better. The worksheet includes written instructions on how to prepare a short oral presentation for the rest of the class.	Students listen and follow the teacher's instructions.
Action and Formulation (adidactical) 15 minutes	The teacher is monitoring group work. Meanwhile, the teacher prepares a PowerPoint presentation consisting of all the diagrams, so the group members can easily explain their topic and argue their conclusions. This is important so that all the students can see all the diagrams and participate in the following discussion.	Students read both graphs, compare them and answer the given questions. They prepare a short oral presentation. Students don't need to present the answers to the given questions, but they should comment on which conclusions are easier to draw from the left or the right graph.

Validation (adidactical) 10 minutes	At this moment, the teacher does not comment but pays attention to reflect on their arguments later during the institutionalisation.	One student from each group comes in front of the class to present their conclusions (groups having the same task come one after another). They comment on different representations of the same data.
Institutionalisation (didactical) 5 minutes	The teacher reflects on the groups' arguments and emphasizes how both representations are accurate but on different scales. And both are "good" depending on what questions you want to answer. *If students did not draw earthquake graphs discreetly emphasize that earthquakes are not a continuous event, so the graph cannot be continuous either.* To show where logarithms are in these examples present the slide with formulas for earthquake magnitude, pH scale and sound level and finish with a statement "*We think linearly, but our senses are on a logarithmic scale.*"	Students listen and take notes. *Recommended* *Get anonymous student feedback by asking two simple questions:* - *What did you learn?* - *What surprised you?*

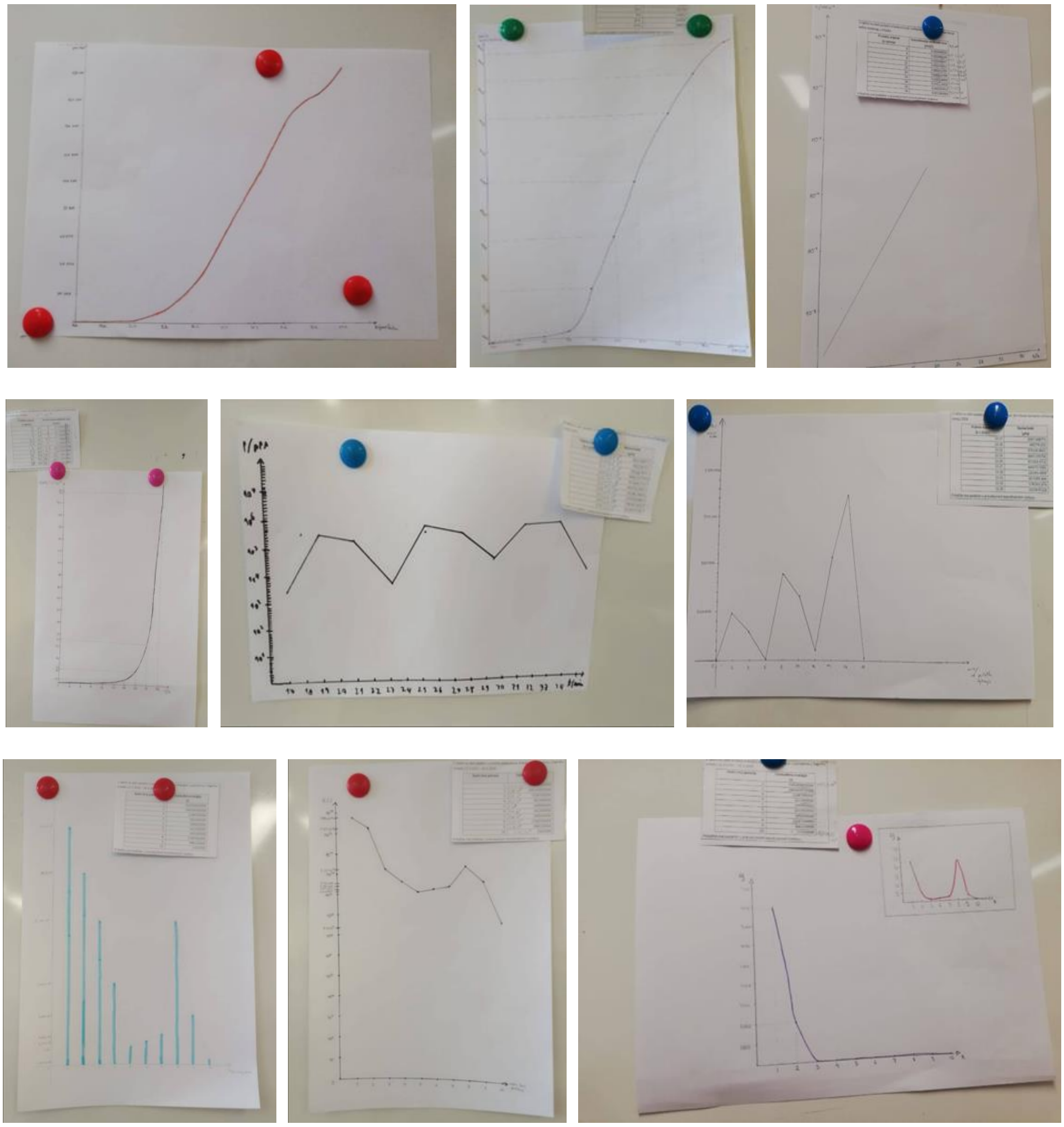
Possible ways for students to realize target knowledge

Students are not expected to use ICT. Since this is only the introduction to the logarithmic function there is a lot to study further. Firstly, what a logarithm is, then how logarithmic equations and inequalities are solved, and in the end how logarithms are used. The concept of logarithm and the logarithmic scale goes beyond the subject of mathematics. Examples already considered have pointed to some applications in physics, chemistry, and biology. In addition to the natural sciences, the logarithmic scale can be of great use in a variety of studies operating with very large or very small numbers. Concepts studied in this lesson will be used as a good reference point in further lessons including logarithms or graphical representations of data (statistics).

In the first action phase, students might

- at first, the paper is turned landscape, the origin is in the middle of the paper, the axes are named x and y, the unit is the same on both axes, and the scale is linear
- draw only the first quadrant for noticing all the data are positive
- turn the paper portrait to fit more data using a linear scale
- sort y-values by value
- find the minimum and maximum y-values
- write y-values in scientific notation
- change the unit prefix of y-values
- draw y-values on a "random" scale only noting which value is greater and not by how much
- draw y-values on a linear by beginning from the smallest value
- draw y-values on a logarithmic scale (powers of 10)
- zooming a part of a graph to better show the difference
- not draw all the data, but instead, note how far from the paper that point would be

EXAMPLES OF STUDENTS' DIAGRAMS



In the second action phase, students might

- validate their representations of data as accurate or not
- discuss the difference between the two given representations of the same data
- answer the questions on the worksheet using the better representation for that purpose
- discuss what to conclude in the final oral presentation

EXAMPLES OF STUDENTS' CONCLUSIONS

- Both representations are good, depending on the information you seek.
- Linear scale gives us a better picture of the actual data but it's hard to put all the data in.
- Logarithmic scale solves the problem of putting different scale data on one graph, but it can be misleading.
- On a linear scale we see the absolute change, on a logarithmic scale we see the rate of change.
- We must pay great attention to what is written on the axes.

NOTE: See the project's webpage for the following editable materials in Word, and the PowerPoint presentation.

HANDOUTS

The table provides data on the total number of those infected in Italy during quarantine in the spring of 2020.

Date (2020.)	Total number of infected
17.2.	3
24.2.	229
2.3.	2038
9.3.	9179
16.3.	27997
23.3.	63941
30.3.	101723
9.4.	143612
16.4.	168932
23.4.	189957

Graph this data in a coordinate system.

The table provides data on the concentration of hydrogen ions during the fermentation of kefir added to milk.

Elapsed time (in hours)	Concentration of hydrogen ions (mol/L)
0	0.000000095
4	0.000000240
8	0.000000437
12	0.000000912
16	0.000001995
20	0.000015488
24	0.000028840
28	0.000114815
32	0.000354813
36	0.001584893

Graph this data in a coordinate system.

The table provides data on the noise level during the 20 minutes of the concert held in July 2020.

Time (9.7.2020.)	Noise level (μPa)
22:17	2907.56
22:19	493776.02
22:21	275124.96
22:23	6645.53
22:25	911024.37
22:27	643472.50
22:29	101281.46
22:31	1075300.86
22:33	1782501.87
22:35	23228.97

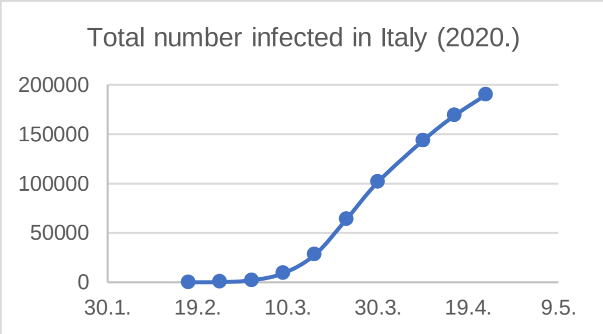
Graph this data in a coordinate system.

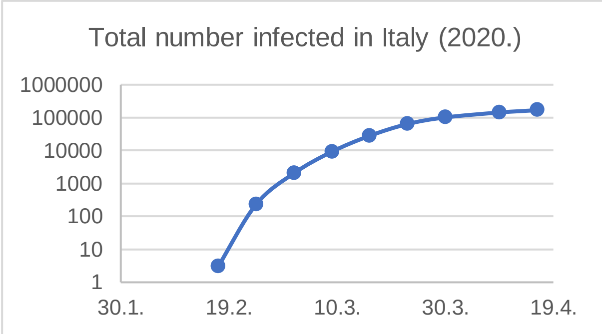
The table provides data on the amount of energy released in the earthquakes in Zagreb between 22. 3. 2020 and 26. 3. 2020.

Earthquake number	Released energy (J)
1.	7943300000000
2.	1995300000000
3.	22387002541
4.	5623400000
5.	1000000000
6.	1412500000
7.	1995300000
8.	22387591200
9.	3981100000
10.	31623000

Graph this data in a coordinate system.

COVID-19 IN ITALY



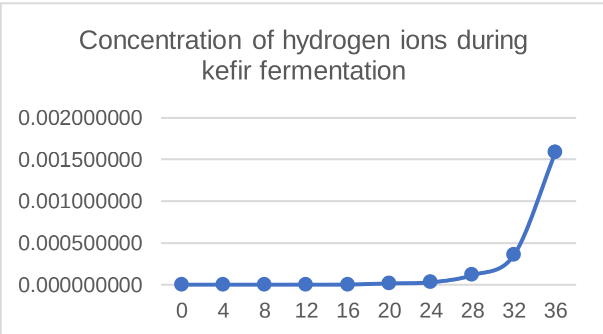


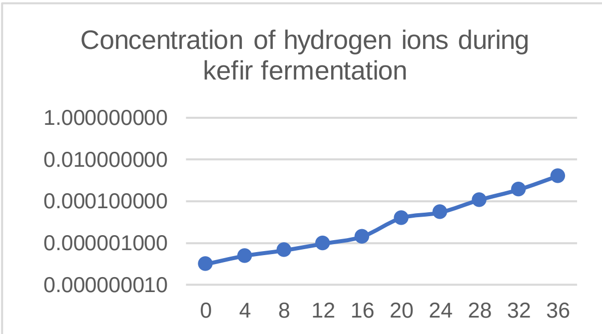
Answer the following three questions as a group.

1. Estimate the number of infected on 30. 4. 2020.
2. Between which dates did the biggest change in the number of patients occur?
3. At what point does the growth of the number of patients slow down?

Then prepare a short presentation for the rest of the class in which you will explain what you concluded from the given graphs, which conclusions were easier to draw from the left, which from the right graph and which of the graphs offers you better information. You do not need to answer questions 1-3 during the presentation.

KEFIR FERMENTATION



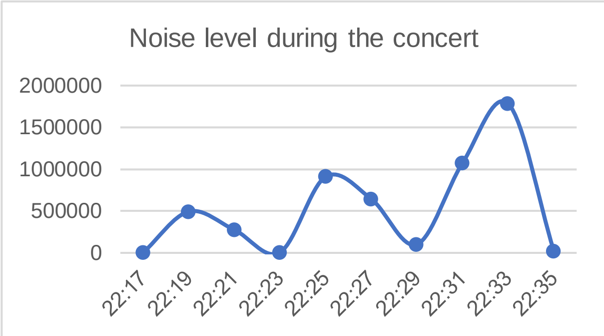


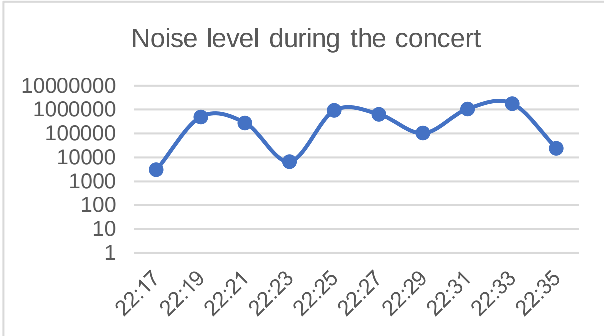
Answer the following three questions as a group.

1. Estimate the concentration of hydrogen ions in the 40^{th} hour.
2. Between which two moments did the biggest change in the concentration of hydrogen ions occur?
3. Concentrations of hydrogen ions above 0.003 prevent fermentation, lumps of kefir may dissolve, and other, undesirable, bacteria may appear. After what fermentation time can kefir cause poisoning?

Then prepare a short presentation for the rest of the class in which you will explain what you concluded from the given graphs, which conclusions were easier to draw from the left, which from the right graph and which of the graphs offers you better information. You do not need to answer questions 1-3 during the presentation.

NOISE AT THE CONCERT



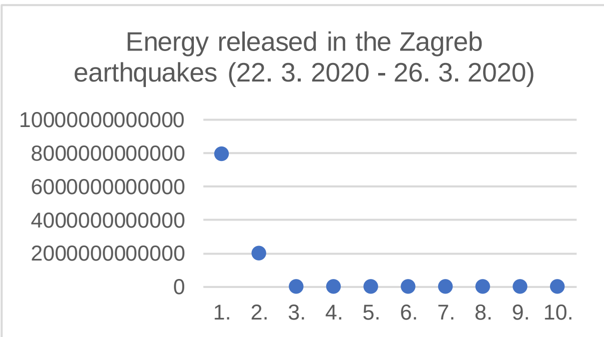


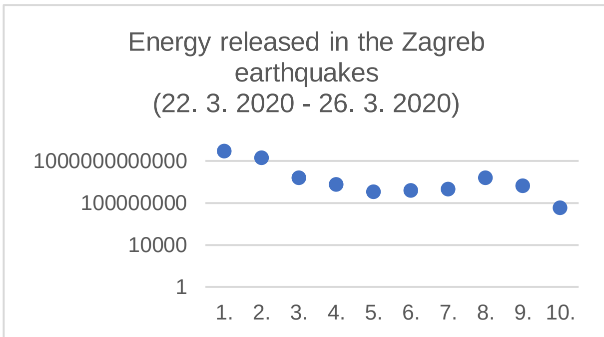
Answer the following three questions as a group.

1. The level of normal conversation is around 6500 μPa. Could a normal conversation have taken place at times during the concert?
2. Between which two moments did the biggest change in noise level occur?
3. Prolonged exposure to noise levels above 200000 μPa may cause hearing damage. How long during these 20 minutes were visitors exposed to that noise level?

Then prepare a short presentation for the rest of the class in which you will explain what you concluded from the given graphs, which conclusions were easier to draw from the left, which from the right graph and which of the graphs offers you better information. You do not need to answer questions 1-3 during the presentation.

ZAGREB EARTHQUAKES





Answer the following three questions as a group.

1. Was more energy released in the 5th earthquake or the 8th earthquake?
2. Between which two of the earthquakes shown is the largest change in the amount of energy released?
3. Most people will feel earthquakes in which the amount of energy released is greater than 63095730000 J. Which of the earthquakes shown did most Zagreb residents feel?

Then prepare a short presentation for the rest of the class in which you will explain what you concluded from the given graphs, which conclusions were easier to draw from the left, which from the right graph and which of the graphs offers you better information. You do not need to answer questions 1-3 during the presentation.

Distance and angle

Constructing models and definitions for basic spatial concepts

Team: V. gymnasium - 1, Zagreb, Croatia

Target knowledge	Defining basic terms from geometry in 3D space: distance from a point to a plane; angle between a line and a plane; angle between two planes.
Broader goals	Understanding written definitions. Making an appropriate manipulative model based on a given definition. Real-world problem-solving.
Prerequisite mathematical knowledge	Distinguishes the terms point, line, and plane in spatial geometry. Analyses and explains mutual positions of points, lines, and planes in space.
Grade	Age 15-16 (2nd grade in Croatia)
Time	80 - 90 minutes
Required material	Worksheets printed on A3 paper. Markers. Different kinds of materials for students to make models: cardboard, styrofoam balls, different size skewers, glue, rope, scissors, etc. Board magnets.

Problem:

Group 1a
What is the distance between the projector lens and the projection screen?
How would you measure that distance?
Based on this example, try to define generally the distance from a point to a plane.

Group 1b
What is the distance between the light bulb and the floor?
How would you measure that distance?
Based on this example, try to define generally the distance from a point to a plane.

Group 2a
What is the angle between your chair leg and the floor?
How would you measure that angle?
Based on this example, try to define generally the angle between a line and a plane.

Group 2b
What is the angle between the paper and the pen you are writing with?
How would you measure that angle?
Based on this example, try to define generally the angle between a line and a plane.

Group 3a
Open your notebook at a 60˚ angle.
How would you measure that angle?
Based on this example, try to define generally the angle between two planes.

Group 3b
Open the windowpane at a 30˚ angle.
How would you measure that angle?
Based on this example, try to define generally the angle between two planes.

Phase	Teacher's actions, including instructions	Students' actions and reactions
Devolution (didactical) 5 minutes	The teacher divides students into six groups - first, the whole class is divided in half (a-half and b-half), and then each half into three groups. Ideally, each group consists of four students. Each group is given an A3 Worksheet 1 leading them to define a term on their own based on real-life examples from immediate surroundings. The a-half works on problems 1a, 2a, and 3a, while the b-half works on problems 1b, 2b, and 3b so that one group from each half of the class works on the same term.	Students listen, follow the teacher's instructions and form groups. They read the assignments and ask questions if something is not clear.
Action (adidactical) 15 minutes	The teacher observes group work and reacts if the students try to do calculations instead of the actual task (of defining the term).	Students visualize the problem and try to write the appropriate definition.
Devolution (didactical) 2 minutes	The teacher instructs groups on how to exchange the worksheets: groups in each half of the class need to exchange their Worksheets 1 in a way that every group gets a new term (e.g., by giving their worksheet to the group on the left and receiving a worksheet from a group on the right). Each group now needs to read the new task and review the definition the other group wrote. They need to think if the definition needs adjustments and try to make it "better". They must write the new and improved definition on Worksheet 2 given to them.	Students listen and follow the teacher's instructions.
Action (adidactical) 15 minutes	The teacher observes group work and stops the students from mentioning the given example from immediate surroundings in their definition if they do so.	Students read the definition and make needed adjustments after the discussion. They write the new and improved definition on a given A3 Worksheet 2.
Devolution (didactical) 2 minutes	The teacher instructs groups on how to exchange worksheets: groups in each half of the class need to exchange their Worksheets 2 in a way that every group gets a new term (e.g., by giving their worksheet to the group on the left and receiving a worksheet from a group on the right). Every group is now introduced to all three terms.	Students listen and follow the teacher's instructions.

	The final task for the groups is to make the manipulative model based solely on the definition that they received from another group using the provided materials. *It is essential to bring to class different kinds of materials: cardboard for representing planes, styrofoam balls as points, different size skewers for lines, glue, rope, scissors, etc.*	
Action (adidactical) 20 minutes	The teacher observes group work and prepares magnets to hang the Worksheets 2 and formal definitions on the board.	Students read the given definition and after a discussion make a manipulative model based on it.
Formulation (adidactical) 3 x 6 minutes	*We repeat this part for every term.* When two groups with the same term are done presenting, the teacher hangs the prepared formal definition on the board.	*We repeat this part for every term.* First one student from each of the two groups with the same term comes in front of the class to present the given definition and the model they made based on it. They hang the given definition on the board with magnets and explain it using a model they have made.
Validation (didactical/ adidactical) 3 x 3 minutes	*We repeat this part for every term, right after hanging the formal definition of that term on the board.* The teacher leads the discussion about the formulation of three definitions on the board: Are students' definitions describing the given term? Is the model consistent with the definition? Are there some deficiencies? Are we missing some special cases? Can we make the definitions clearer?	Students compare the formal definition with their own and participate in the discussion. All students can equally engage since every student worked on all three terms in some phase.
Institutionalisation (didactical) 3 minutes	The teacher comments on students' work. Accentuate the importance of appropriate use of mathematical language and precise definitions. No less important is the beauty of simplicity.	Students write down the final version of the definitions in their notebooks. *Recommended: get anonymous student feedback by asking two simple questions:* - *What did you learn?* - *What surprised you?* *(using one of many digital tools for this purpose or just the good old pen and paper method).*

Possible ways for students to realize target knowledge

Students are not expected to use ICT.

In the first action phase, students might
- look at the real example from immediate surroundings given to them on Worksheet 1, spin it around and make some conclusions based on observation,
- make sketches,
- discuss defining the term and how to write down the definition,
- calculate the task given as an example which we do not expect them to solve.

In the second action phase, students might
- look at the real example from immediate surroundings given to them on Worksheet 1 received from another group, spin it around and make some conclusions based on observation,
- look at sketches the other group has made and make adjustments,
- discuss the definition given to them, do they understand it, does it define the term in question, does it need improvement in the sense of clarity, using appropriate mathematical language or simplifying,
- discuss how to write a definition on Worksheet 2.

In the third action phase, students might
- read the definition they got from another group and try to understand it,
- look at the materials the teacher brought for students to make models,
- discuss how to make a model of a given term based on a given definition using materials provided,
- make manipulative models.

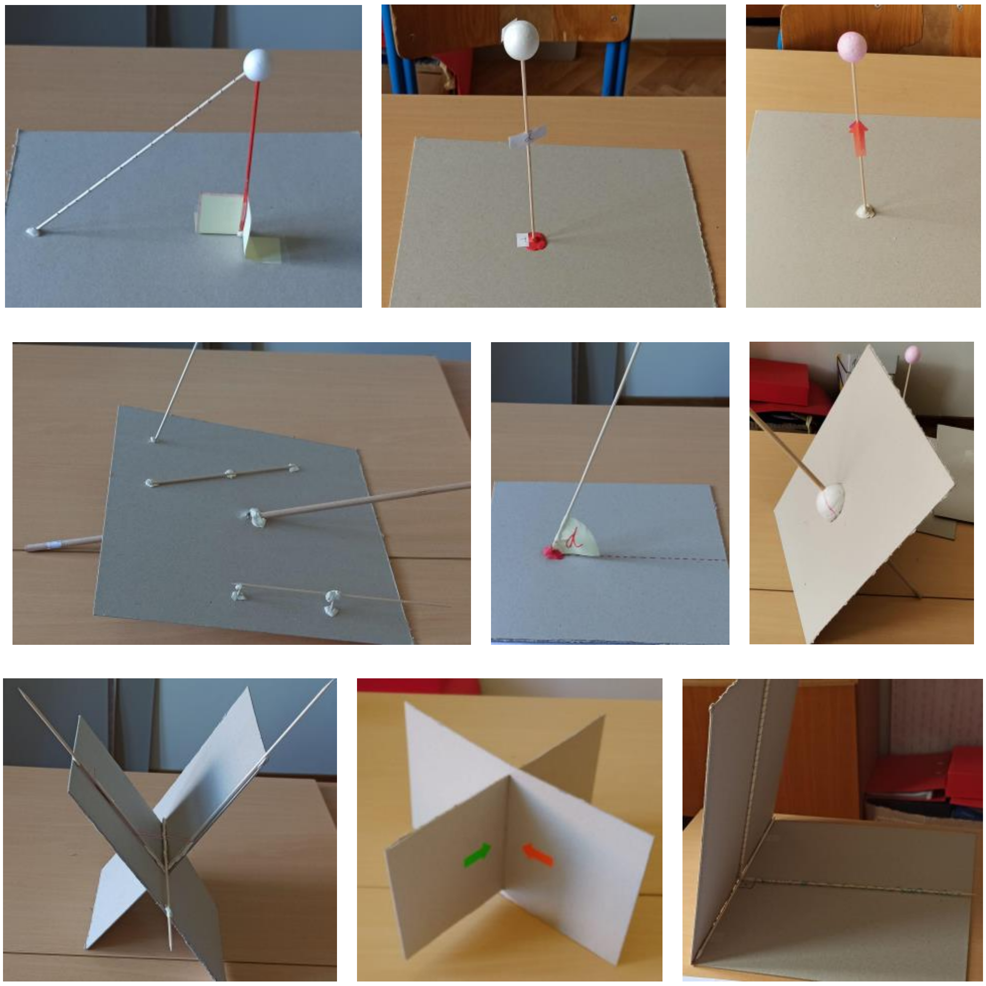
EXAMPLES OF FORMULATIONS OF STUDENTS' DEFINITIONS

DISTANCE FROM A POINT TO A PLANE
- is the distance of the point from its orthogonal projection on the plane.
- is the length of a segment from the point to its orthogonal projection on the plane.
- is the length of the segment perpendicular to that plane where one end of that segment is on the plane and the other is the point whose distance from the plane we observe.
- is the length of the shortest segment that connects that point and that plane.

ANGLE BETWEEN A LINE AND A PLANE
- is the angle between the line and its orthogonal projection on that plane.
- is the angle that the line closes with the line that is its orthogonal projection on the given plane.

ANGLE BETWEEN TWO PLANES
- is the angle between two lines lying on these planes and are perpendicular to the line in which these two planes intersect.
- is the space between two rays lying on different planes, they are perpendicular to the line of intersection of the planes and their origin is on that line.
- is the angle between the line lying on a plane perpendicular to the intersection line of two planes and the line which is its orthogonal projection on another plane.
- is the angle enclosed by the orthogonal projections of the lines lying on the two planes on the third perpendicular plane.

EXAMPLES OF STUDENTS' MODELS



Further study

The concepts studied in this lesson are going to be used in solving the following problems in spatial geometry where distances and angles in polyhedrons are supposed to be calculated, especially those involving pyramids. Since students usually experience a lot of difficulties in visualizing and drawing sketches of the angle between the base and the lateral face of the pyramid, or between the base and its lateral edge, the expectations are that this lesson is going to help them in determining and calculating wanted angles and distances. It will be used as a good reference point in further lessons when recalling important definitions and polyhedron properties.

NOTE: See the project's webpage for the following editable materials in Word.

WORKSHEETS

What is the distance between the projector lens and the projection screen? How would you measure that distance?

Based on this example, try to define generally the distance from a point to a plane. Write down your definition neatly:

DISTANCE FROM A POINT TO A PLANE

What is the distance between the light bulb and the floor? How would you measure that distance?

Based on this example, try to define generally the distance from a point to a plane. Write down your definition neatly:

DISTANCE FROM A POINT TO A PLANE

DISTANCE FROM A POINT TO A PLANE
is equal to the distance of the point from its orthogonal projection on the plane.

What is the angle between the leg of the chair you are sitting on and the floor? How would you measure that angle?

Based on this example, try to define generally the angle between a line and a plane. Write down your definition neatly:

ANGLE BETWEEN A LINE AND A PLANE

What is the angle between the paper and the pen you are writing with? How would you measure that angle?

Based on this example, try to define generally the angle between a line and a plane. Write down your definition neatly:

ANGLE BETWEEN A LINE AND A PLANE

ANGLE BETWEEN A LINE AND A PLANE
is the angle between the line and its orthogonal projection on the plane.

Open your notebook at a 60° angle. How would you measure that angle?

Based on this example, try to define generally the angle between two planes. Write down your definition neatly:

ANGLE BETWEEN TWO PLANES

Open the windowpane at a 30° angle. How would you measure that angle?

Based on this example, try to define generally the angle between two planes. Write down your definition neatly:

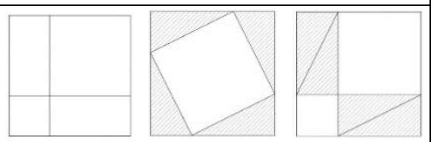
ANGLE BETWEEN TWO PLANES

ANGLE BETWEEN TWO PLANES
is the angle between the perpendiculars to the intersection of these planes each belonging to one of the planes, both passing at the same point on the intersection of the planes.

Argumentation

Discovering algebraic identities in geometric figures

Team: V. gymnasium - 2, Zagreb, Croatia

Target knowledge	Discovering the connection between the well-known algebraic identities and their geometric proofs.
Broader goals	Ability to express one's own reasoning clearly, precisely, and systematically.
Prerequisite mathematical knowledge	Basic algebra of multiplying binomials.
Grade	Age 15-16 (1st grade in Croatia)
Time	60 minutes
Required material	Handouts, white posters (A3 or bigger) for presentation.
Problem: Recognize the mathematical fact behind the obtained drawing.	

Phase	Teacher's actions, including instructions	Students' actions and reactions
Devolution (didactical) 5 minutes	The teacher organizes the students into groups of 3-4 and shares the handouts. One handout per group, different groups may get different handouts. The instruction is very short and perhaps mysterious.	Students consider the handout and will probably be confused by the lack of information. Quickly they get reassured by the teacher that they solve the problem. Students should realize that "everything is allowed".
Action (adidactical) 25 minutes	The teacher observes the students' work without interfering. The teacher reminds the students that the emphasis is on argumentation and hands out white posters for presentation. If the students are stuck only with concrete numbers (measurements), the teacher directs the students to look for more general rules.	Students work in groups. They first discuss ideas to tackle the problem. As they have a very concrete picture in front of them, they might start by measuring. Some students may immediately recognize the formula behind the figure but will take time to write their reasoning.
Formulation (adidactical) 15 minutes	The teacher prepares the board for presentations and invites the students to present their posters. Groups that were working on the same problem present immediately one after another.	Students hang their posters on the board and explain their reasoning and conclusions.
Validation (adidactical) 10 min	For each of the four problems, the teacher invites the students to ask questions and to compare the work of different groups.	Students comment on the different approaches and ask questions. They might share their difficulties and additional explanations.
Institutionalisation (didactical) 5 min	The teacher highlights the formulas behind the pictures and emphasises the argumentation points given by the students.	Students listen and write down notes about the goals concerning the argumentation. Students give feedback on the lesson.

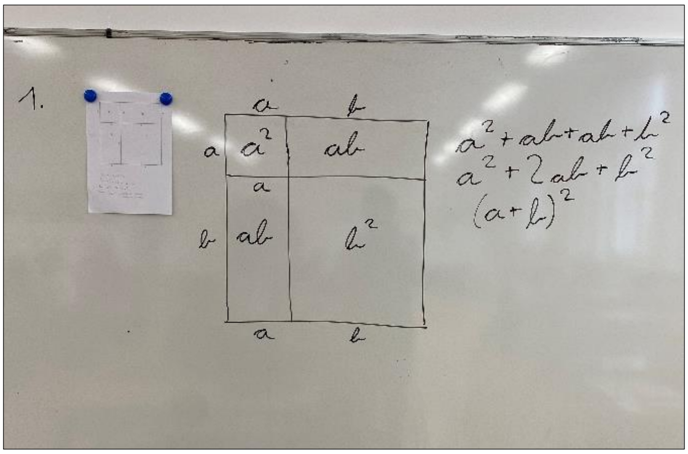
Possible ways for students to realize target knowledge

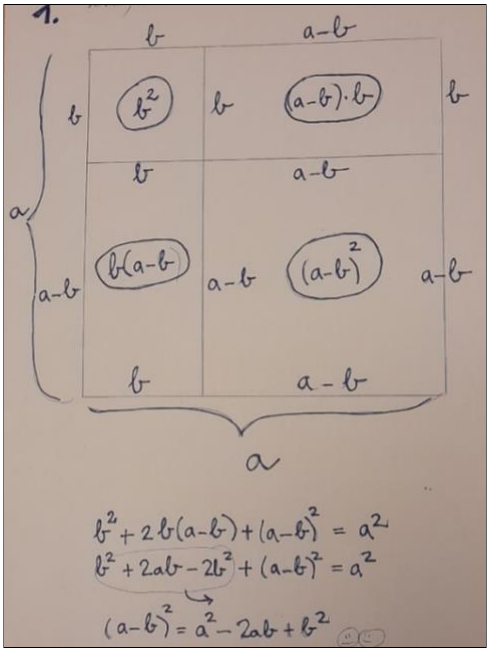
At first, the students will probably just look at the given picture, turn it in various positions, and then start to notice some segments of the same lengths, mark them with the same labels, and then gradually come to the idea of using the area of the rectangle.

Students might observe various symmetries and draw the diagonals of rectangles – they observe and discuss whether they could conclude something from it. One might hear: "*Look, this area is equal to this one, so when we add them up, we will get …*", or: "*That could be the Pythagorean theorem, because we have a right triangle, we need to mark the legs and the hypotenuse and then write it down …*".

Some students might measure the lengths in the given figure and find some arithmetical links between the lengths. Possibly with the teacher's instruction, they might translate these patterns into algebraic rules by replacing numbers with letters (generalized numbers).

- For Handout 1 there are two typical solutions. More prominently, the students will explain the picture with the formula for the square of the binomial sum (left). Other students might explain the picture by the formula for the square of the binomial difference (right).

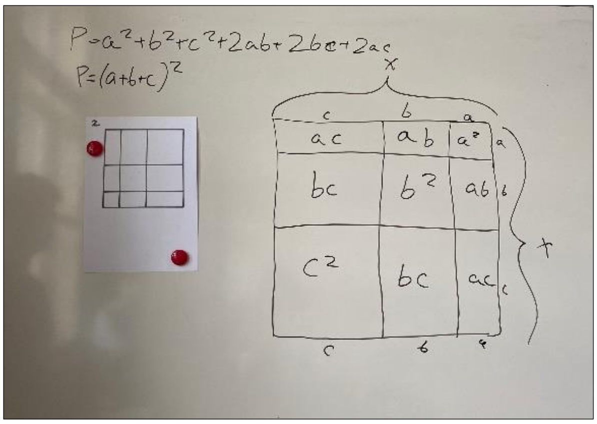




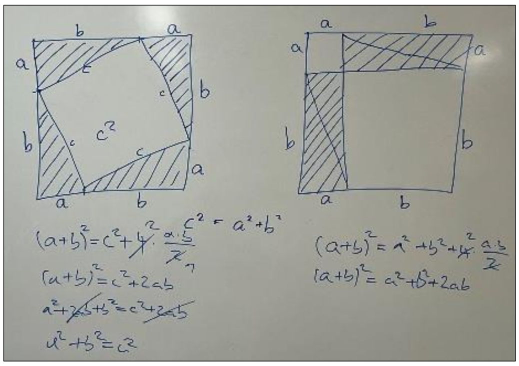
The two ways have the same reasoning at their core, but the use of labels is different, which implies in the second case that more algebraic manipulations are needed to reach a well-known formula.

- For Handout 2 students might need more time. It is crucial that the students put the labels on the sides of the rectangles and write down the area of each of the observed rectangles. The picture can then be explained by collecting the nine areas into a formula for the square of a trinomial:

$$(a + b + c)^2 = a^2 + b^2 + c^2 + 2ab + 2ac + 2bc.$$

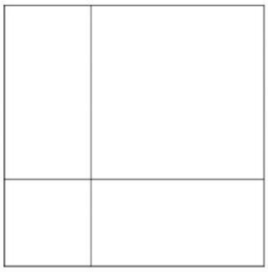
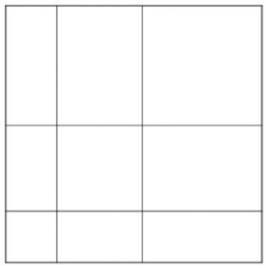
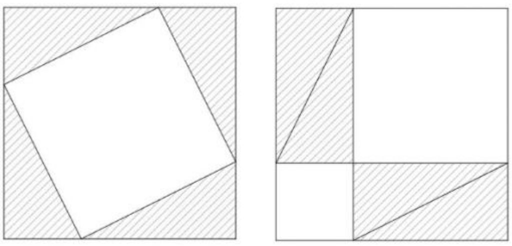
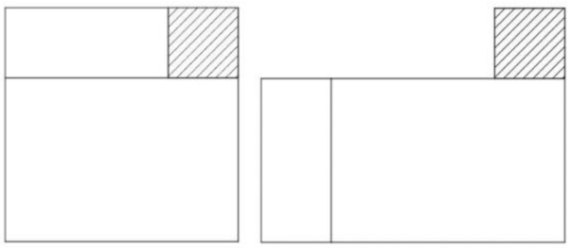


- For Handout 3 it is also important to put the labels on the sides of the four congruent dark triangles.
  Then the pictures give the geometrical proof of Pythagoras' theorem. If a and b are legs and c is the hypotenuse of a right-angled triangle, then
  $$a^2 + b^2 = c^2.$$
  To arrive at this identity, the students actually need to note that the identity corresponds to the white parts of the two pictures, which are of equal area because they are obtained by removing the congruent triangles from the same big square.



- For Handout 4 there is one 'aimed' solution, while students might find more explanations. One way is to observe that once the shaded square is cut out of the big square, the remaining area can be cut into two rectangles which can be moved to form one rectangle. This leads to the "difference of the squares" formula
  $$a^2 - b^2 = (a-b)(a+b).$$
  Other students might simply focus only on the figure on the left and draw the extension of the side of the shaded square until they obtain the figure from Handout 1. Their conclusion might be the same as for some students thinking about that handout.

NOTE: See the project's webpage for the following editable materials in Word.

HANDOUT 1	HANDOUT 2
	
HANDOUT 3	**HANDOUT 4**
	

How does it work?

Understanding the inner working of mathematical induction

Team: XV. gymnasium - 2, Zagreb, Croatia

Target knowledge	Discovering the Principle of Mathematical Induction.
Broader goals	Distinguishing and connecting inductive and deductive reasoning. Reasoning by analogy. Mathematical communication. Reading a mathematical text with understanding. Synthesis.
Prerequisite mathematical knowledge	Manipulation of simple algebraic expressions. The set of natural numbers (positive integers).
Grade	Age 18 (4th grade in Croatia)
Time	90 minutes
Required material	Worksheets, textbook, post-it notes.

Problem:
Investigate the procedure shown in the worksheet. What is the relation between two consecutive rows? Does the statement hold in general for all natural numbers? How can you prove it based on the started reasoning?

$$\frac{2 \cdot (2+1)}{2}$$

Additional problem:
Read the explanation in the textbook and find mistakes in the second worksheet.

Phase	Teacher's actions, including instructions	Students' actions and reactions
Devolution (didactical) 5 minutes	The teacher hands out the Worksheet 1. The proof of the statement might be familiar to some of the students, however, this time the proof will be done differently. Depending on the desired level of guidance, the teacher could modify the questions on the worksheet.	Students listen and consider the worksheet.
Action (adidactical) 15 minutes	The teacher observes the students' work without interfering. The teacher organizes the board according to the questions on the worksheet.	Students work individually, answering questions using the written instructions. Students should notice the connection between two consecutive steps and by analogy discover the idea of how to use this way of reasoning in general.
Formulation (didactical/ adidactical) 5 minutes	The teacher asks the students to present their answers on the board.	Students put their answers written on the sticky notes on the board and then read all the answers.
Validation (didactical/ adidactical) 5 minutes	The teacher emphasizes some of the answers and asks the students to compare and comment.	Students might observe different ways of phrasing and could notice that some of the students have (not) focused on the link between the consecutive rows. They express the main aspects of the Principle of Mathematical Induction in their own words or using analogies such as falling dominoes.

Institutionalisation (didactical) 10 minutes	The teacher summarizes the students' work and directs the students to read the explanation in the textbook.	Students read and compare their explanations with the one given in the textbook. They may copy parts they find important to their notebooks.
Devolution (didactical) 5 minutes	The teacher hands out Worksheet 2. Now, the students need to study a few written proofs, find mistakes and comprehensively discuss the Principle.	Students consider the second worksheet.
Action (adidactical) 20 minutes	The teacher observes the students' work. If the students ask about the mistakes, they think they have found, the teacher will ask for a justification, but not confirm the correct answer in this phase.	Students read the examples, analyse the proofs, and through discussion, justify their choices. Students should notice that one example misses the induction base, and another one assumes the induction step for all natural numbers. The final example is correct, but the base is $n = 3$.
Formulation (didactical/ adidactical) 5 minutes	The teacher invites the students to share their findings. Based on the observations in the previous phase, the teacher selects some of the students to explain their reasoning.	Students share their reasoning about the possible mistakes in the proof. It is expected that they will start using the language of the principle (the basis, the assumption for some n, the step, etc.)
Validation (didactical/ adidactical) 15 minutes	The teacher asks the students to comment on the answers. Particularly, where there is a dispute between students. The teacher also directs the discussion to connect it with the first activity.	Students comment on various answers from other students and revise their answers through the discussion.
Institutionalisation (didactical) 5 minutes	The teacher summarizes the students' work and emphasizes the importance of the steps of the principle.	Students acknowledge the subtle points of applying the method of mathematical induction.

Possible ways for students to realize target knowledge

Worksheet 1. The students are expected to notice that we can check that the statement is true for $n = 1, 2, 3, \ldots$ but we do not know that it holds for all natural numbers n.

The students will write the corresponding formulas for $n = 4$ and $n = 5$:

$$1 + 2 + 3 + 4 = \frac{3 \cdot (3 + 1)}{2} + 4 = \frac{3 \cdot 4 + 4 \cdot 2}{2} = \frac{4 \cdot (4 + 1)}{2}$$

$$1 + 2 + 3 + 4 + 5 = \frac{4 \cdot (4 + 1)}{2} + 5 = \frac{4 \cdot 5 + 5 \cdot 2}{2} = \frac{5 \cdot (5 + 1)}{2}$$

From these, they will see that we could probably check the identity for more natural numbers. It is crucial that the students realize on their own that reasoning by analogy (e.g., "we *obviously see* this will continue for *all* natural numbers") is not good enough to constitute a proof.

For $n = 6$, one could also write out a similar line of reasoning, but if we wonder about a greater number, we will need a different kind of reasoning. Students should observe how two consecutive rows are connected: they should also mark the result obtained for $n = 4$ in the line of reasoning for $n = 5$.

$$1+2+3+4=\frac{3\cdot(3+1)}{2}+4=\frac{3\cdot 4+4\cdot 2}{2}=\frac{4\cdot(4+1)}{2}$$

$$1+2+3+4+5=\frac{4\cdot(4+1)}{2}+5=\frac{4\cdot 5+5\cdot 2}{2}=\frac{5\cdot(5+1)}{2}$$

Next, the students conclude maybe the most important part of the reasoning: what we can prove in general is the link between the two consecutive numbers. If we know that, for example, the statement holds for $n=50$, using that fact, we can prove the statement for $n=51$. This is true for any pair of consecutive natural numbers n and $n+1$. If we know that the result for n is

$$1+2+\cdots+n=\frac{n\cdot(n+1)}{2},$$

then we can reason that

$$1+2+\cdots+n+(n+1)=\frac{n\cdot(n+1)}{2}+n+1=\frac{n\cdot(n+1)+(n+1)\cdot 2}{2}=\frac{(n+1)\cdot(n+2)}{2}.$$

In the end, there should be a formulation of the Principle of Mathematical Induction:

If a statement about natural numbers holds for $n=1$, and we know can prove that it holds for $n+1$ using that it holds for n, then it holds for any natural number.

Or using symbols:

If $T_1 \;\wedge\; (\forall n\in\mathbb{N})(T_n\Rightarrow T_{n+1})$, then $(\forall n\in\mathbb{N})\; T_n$.

Worksheet 2.

After reading about the principle from the textbook, the students should be able to describe the principle more confidently and, based on the examples, argue why the basis and the step are important.

In Example 1, there is no basis for mathematical induction. In this example, we see that the step might be true, but the statement actually does not hold.

In Example 2, there is an assumption that the statement holds for all natural numbers. It might help to compare the statement of the principle written in symbols to the structure of the proof in this example, which we might write as:

If $T_1 \;\wedge\; \big(((\forall n\in\mathbb{N})T_n)\Rightarrow T_{n+1}\big)$, then $(\forall n\in\mathbb{N})\; T_n$.

It should be clear from this example that it makes no sense to assume what we want to prove.

In Example 3, there is no logical mistake. It shows that the basis does not need to be $n=1$. In that case, we conclude that the statement is true for all natural numbers not smaller than the number used in the basis. In symbols we can write:

If $T_{n_0} \;\wedge\; (\forall n\in\mathbb{N}, n\geq n_0)(T_n\Rightarrow T_{n+1})$, then $(\forall n\in\mathbb{N}, n\geq n_0)\; T_n$.

NOTE: See the project's webpage for the following editable materials in Word.

WORKSHEET 1

We want to prove that statement **T** holds for all natural numbers n.

$$\textbf{T}: 1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}$$

Investigate the following procedure. Write the answers to the questions below on separate sticky notes.

$n = 1$	Check: $1 = \frac{1 \cdot (1+1)}{2}$
$n = 2$	$1 + 2 = \frac{1 \cdot (1+1)}{2} + 2 = \frac{2 + 2 \cdot 2}{2} = \frac{2 \cdot (2+1)}{2}$
$n = 3$	$1 + 2 + 3 = \frac{2 \cdot (2+1)}{2} + 3 = \frac{2 \cdot 3 + 3 \cdot 2}{2} = \frac{3 \cdot (3+1)}{2}$
$n = 4$	
$n = 5$	

1. What can you conclude, for which numbers the statement **T** holds?
2. Does it hold for $n = 6$? On what grounds can you conclude that?
3. What is the relation between steps in the procedure for $n = 4$ and $n = 5$?
4. Can you connect steps for $n = 50$ and $n = 51$? Write that down.
5. How are generally any two consecutive lines connected?
6. Can you prove this way that a statement holds for any natural number?

Less guided alternative:

Investigate the following procedure. What is the relation between two consecutive rows? Does the statement hold in general for all natural numbers? How can you prove it based on the started reasoning?

An explanation of the Principle of Mathematical Induction and a worked-out example from a Croatian textbook (authors are project members).

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Matematička indukcija način je dokazivanja da neka tvrdnja vrijedi za sve prirodne brojeve n počevši od 1 ili od nekoga prirodnog broja n_0. Princip matematičke indukcije sastoji se od dvaju osnovnih dijelova, **baze indukcije** i **koraka indukcije**.

Princip dokazivanja matematičkom indukcijom

Neka je $T(n)$ tvrdnja koja ovisi o broju n.

I. **(Baza indukcije)** Tvrdnja $T(n)$ vrijedi za $n = 1$ (ili $n = n_0$)

II. **(Korak indukcije)** Iz pretpostavke da tvrdnja $T(n)$ vrijedi za **neki** proizvoljni prirodni broj n slijedi da tvrdnja vrijedi i za sljedeći prirodni broj $n + 1$.

Ako je ispunjeno I. i II., tvrdnja $T(n)$ vrijedi za sve prirodne brojeve n (ili sve prirodne brojeve veće od n_0.

Pojasnimo zašto, kada provjerimo bazu i provedemo korak indukcije, tvrdnja vrijedi za svaki prirodni broj n.

Jednostavno, ako tvrdnja vrijedi za $n = 1$, tada zbog koraka indukcije vrijedi i za sljedeći prirodni broj, za $n = 2$. Zatim zbog koraka indukcije vrijedi i za sljedeći prirodni broj, za $n = 3$. Kako vrijedi za $n = 3$, vrijedi i za $n = 4$ i tako dalje, ovaj se proces ne zaustavlja, što znači da će tvrdnja vrijediti za sve prirodne brojeve.

Vrlo se često ovaj princip ilustrira uz pomoć „domino-efekta" ili principa rušenja beskonačnoga broja domino-pločica. Sve će se pločice srušiti ako su ispunjeni sljedeći uvjeti:

I. Prva će se pločica srušiti.

II. Ako se sruši bilo koja pločica u nizu, ona će svojim rušenjem srušiti i sljedeću pločicu.

Sve će se pločice srušiti jer će se prva srušiti zbog I., ona će srušiti drugu zbog II., druga će srušiti treću, opet zbog II., treća četvrtu, opet zbog II. itd.

PRIMJER

1. Dokažimo matematičkom indukcijom da tvrdnja 1 vrijedi za sve prirodne brojeve n.

$T(n): 2+5+8+11+\ldots+(3n-1)=\frac{n\cdot(3n+1)}{2}$.

Rješenje:

I. Baza indukcije.

Provjerimo tvrdnju za $n = 1$.

$3\cdot 1-1=\frac{1\cdot(3\cdot 1+1)}{2}$, što daje vrijednost 2 na obje strane jednakosti, pa je tvrdnja točna.

II. Korak indukcije.

Pretpostavimo da $T(n)$ vrijedi za neki proizvoljni prirodni broj n:

$2+5+8+11+\ldots+(3n-1)=\frac{n\cdot(3n+1)}{2}$.

Koristeći se ovom pretpostavkom, trebamo dokazati da tvrdnja vrijedi i za sljedeći prirodni broj, odnosno da vrijedi $T(n + 1)$:

$2+5+8+11+\ldots+(3n-1)+\left(3(n+1)-1\right)=\frac{(n+1)\cdot(3(n+1)+1)}{2}$.

Desnu stranu od $T(n + 1)$ možemo zapisati kao $\frac{(n+1)\cdot(3n+4)}{2}=\frac{3n^2+7n+4}{2}$.

Na lijevoj strani primijenit ćemo pretpostavku za zbroj prvih n pribrojnika i pridodati $(n + 1)$. član.

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$$\underbrace{2+5+8+11+\ldots+(3n-1)}_{\text{po pretpostavci}=\frac{n(3n+1)}{2}}+3(n+1)-1=\frac{n(3n+1)}{2}+3n+2=\frac{n(3n+1)+6n+4}{2}=$$

$$=\frac{3n^2+7n+4}{2}$$

Nakon sređivanja, dobili smo isti izraz kao i na desnoj strani tvrdnje $T(n + 1)$, čime smo pokazali da za proizvoljni n iz pretpostavke $T(n)$ slijedi $T(n + 1)$. Zaključujemo, tvrdnja $T(n)$ vrijedi za sve prirodne brojeve n.

WORKSHEET 2

Consider the three examples below and discuss if the proofs are correct.

Based on the examples answer the following questions:

1. Describe the Principle of Mathematical Induction in your own words.
2. Why do we need the induction step?
3. Why do we need both the induction step and the basis?
4. Why is the induction basis necessary?

Example 1:
Prove that the statement holds for all natural numbers n:

$$2 + 4 + 6 + \cdots + 2n = n^2 + n + 2. \qquad (*)$$

Proof:
We assume that the statement holds for any given natural number n:

$$2 + 4 + 6 + \cdots + 2n = n^2 + n + 2$$

Now we will prove that it holds for the next natural number $n + 1$, that is

$$2 + 4 + 6 + \cdots + 2n + 2(n + 1) = (n + 1)^2 + (n + 1) + 2$$

From the LHS:

$$\underbrace{2 + 4 + 6 + \cdots + 2n}_{\text{assumption } n^2+n+2} + 2(n + 1) = n^2 + n + 2 + 2(n + 1) = (n^2 + 2n + 1) + (n + 1) + 2$$
$$= (n + 1)^2 + (n + 1) + 2.$$

Hence $(*)$ is proved.

Example 2:
Prove that

$$\frac{1}{1 \cdot 3} + \frac{1}{3 \cdot 5} + \cdots + \frac{1}{(2n - 1) \cdot (2n + 1)} = \frac{n}{2n + 1} \ (**)$$

holds for all natural numbers n.

Proof:

I. Induction base
For $n = 1$ the statement holds because $\frac{1}{1 \cdot 3} = \frac{1}{2 \cdot 1 + 1}$.

II. Induction step
Let's assume that $(**)$ holds for all natural numbers n:

$$\frac{1}{1 \cdot 3} + \frac{1}{3 \cdot 5} + \cdots + \frac{1}{(2n - 1) \cdot (2n + 1)} = \frac{n}{2n + 1}$$

Using the assumption, we must prove that the statement also holds for the next natural number $n + 1$:

$$\frac{1}{1\cdot 3}+\frac{1}{3\cdot 5}+\cdots+\frac{1}{(2n-1)\cdot(2n+1)}+\frac{1}{(2(n+1)-1)\cdot(2(n+1)+1)}=\frac{n+1}{2(n+1)+1}=\frac{n+1}{2n+3}.$$

From the LHS:

$$\begin{aligned}\frac{1}{1\cdot 3}+\frac{1}{3\cdot 5}+\cdots+\frac{1}{(2n-1)\cdot(2n+1)}+\frac{1}{(2(n+1)-1)\cdot(2(n+1)+1)} &= \{\text{using the assumption}\}\\ =\frac{n}{2n+1}+\frac{1}{(2n+1)(2n+3)}=\frac{n(2n+3)+1}{(2n+1)(2n+3)}=\frac{2n^2+3n+1}{(2n+1)(2n+3)}\\ =\frac{2n^2+2n+n+1}{(2n+1)(2n+3)}=\frac{2n(n+1)+(n+1)}{(2n+1)(2n+3)}=\frac{(n+1)(2n+1)}{(2n+1)(2n+3)}=\frac{n+1}{2n+3}\end{aligned}$$

.

Hence, the statement $(**)$ holds for all natural numbers.

Example 3:
Let's prove that for all natural numbers $n \geq 3$

$$1\cdot 3+2\cdot 4+3\cdot 5+\cdots+(n-2)\cdot n=\frac{(n-2)(n-1)(2n+3)}{6}.$$

Proof:

I. Induction base

For $n=3$ the statement holds because $1\cdot 3=\frac{(3-2)(3-1)(6+3)}{6}=3$.

II. Induction step

Assume that the statement holds for a given natural number $n \geq 3$:

$$1\cdot 3+2\cdot 4+3\cdot 5+\cdots+(n-2)\cdot n=\frac{(n-2)(n-1)(2n+3)}{6}.$$

Using the assumption, we have to prove that the statement holds for the next natural number $n+1$:

$$\begin{aligned}&1\cdot 3+2\cdot 4+3\cdot 5+\cdots+(n-2)\cdot n+(n-1)\cdot(n+1)\\ &=\frac{\big((n+1)-2\big)\big((n+1)-1\big)(2(n+1)+3)}{6}=\frac{(n-1)n(2n+5)}{6}.\end{aligned}$$

From the LHS:

$$\begin{aligned}&1\cdot 3+2\cdot 4+3\cdot 5+\cdots+(n-2)\cdot n+(n-1)\cdot(n+1)=\{\text{using assumption}\}\\ &=\frac{(n-2)(n-1)(2n+3)}{6}+(n-1)(n+1)\\ &=\frac{(n-2)(n-1)(2n+3)+6(n-1)(n+1)}{6}\\ &=\frac{(n-1)(2n^2+3n-4n-6+6n+6)}{6}=\frac{(n-1)(2n^2+5n)}{6}=\frac{(n-1)n(2n+5)}{6}.\end{aligned}$$

Hence, the statement holds for any natural number $n \geq 3$.

The radian-meter

Defining trigonometric values

Team: XV. gymnasium - 2, Zagreb, Croatia

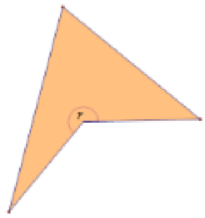
Target knowledge	Expanding the definition of the sine (cosine) of a number to the definition of the sine (cosine) of an angle.
Broader goals	Mathematical communication. Distinguishing and connecting the sine of a number and the sine of an angle. Application of properties of trigonometric functions of numbers to angles.
Prerequisite mathematical knowledge	Trigonometric functions of numbers – the unit circle.
Grade	Age 17 (3rd grade in Croatia)
Time	40 minutes
Required material	Worksheets, radian-meter (a circle with arc measures), figures cut out of paper.

Problem:

You are given a circle with arc measures and some figures cut out of paper.

a) Using the given transparencies, determine approximately the sine and the cosine of the marked angles in the two triangles and one quadrilateral.

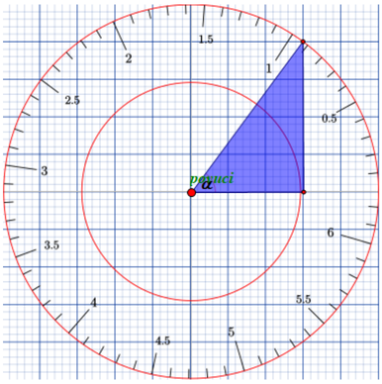
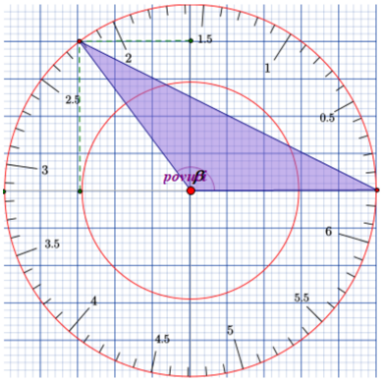
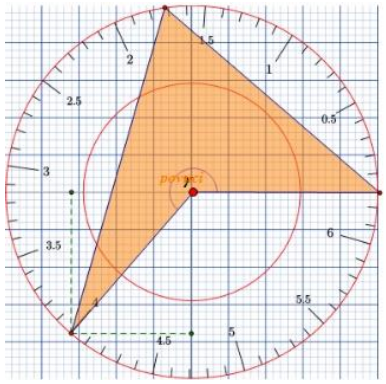
b) Express the length of the marked side of the triangle using the trigonometric values of the marked angle of the triangle.



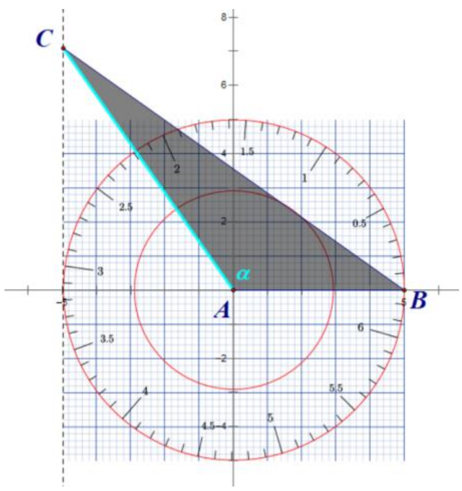
Phase	Teacher's actions, including instructions	Students' actions and reactions
Devolution (didactical) 2 minutes	The teacher presents the first problem to the students.	Students listen and eventually ask questions if they do not understand something.
Action and formulation (adidactical) 10 minutes	The teacher observes the students' work without interfering.	Students work individually, trying to find the solution. They place the triangle on the circle in the appropriate position and think about how to read the sine and cosine of the marked angles.
Validation (didactical) 5 min	For each of the three cases, the teacher invites one student to present his / her method and solution.	Students present the results and describe their way of thinking.
Devolution (didactical) 5 minutes	The teacher poses the second problem.	Students listen and ask questions if they do not understand the task.
Action and formulation (adidactical) 8 minutes	The teacher observes the students' work.	Students connect the task with the previous problem and apply it to the triangle to calculate the unknown side of the triangle.
Validation (didactical) 5 min	One or two students present their solutions - the teacher leads the discussion.	Students listen and discuss.
Institutionalisation (didactical) 5 min	Based on students' results, the teacher summarizes how the definition of the sine of an angle coincides with the definition of the sine of a number.	Students listen, ask questions, and take notes.

Possible ways for students to realize target knowledge

In the first activity, students are expected to place the figures in a position that will allow them to read the sine and cosine of marked angles. They try to define the measure of the angle in radians.

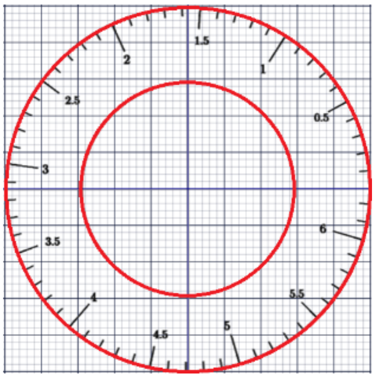
In the second activity, the students should express the length of the marked side in a triangle by the trigonometric values of the marked angle. After identifying the x-coordinate of point C as -1, students can observe the rectangular triangle on the left with side AC as the hypotenuse.



The angle in this triangle is supplementary to angle α and they know that their cosine values are opposite, therefore $\cos(180° - \alpha) = -\cos\alpha$. Finally, $|AC| = -\frac{1}{\cos\alpha}$.

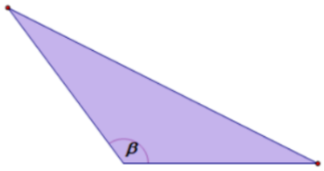
NOTE: See the project's webpage for the following editable materials in Word.

Radian-meter:

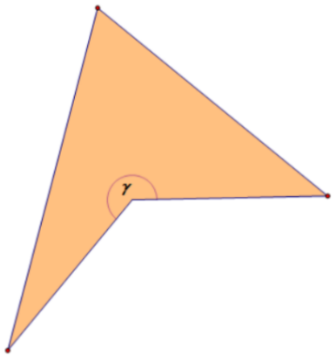


Right-angled triangle with a hypotenuse having length of 1 unit:

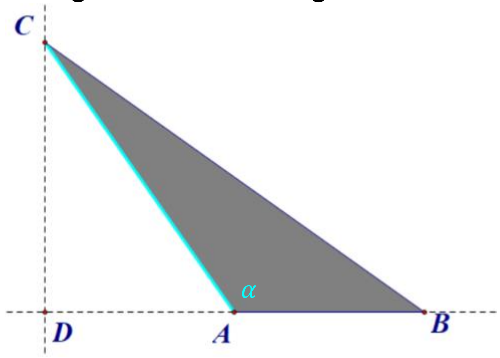
Isosceles triangle with legs having a length of 1 unit:



Concave quadrilateral whose two sides, that form a reflex angle, have a length of 1 unit:



Triangle with a base length of 1 unit and the distance AD also 1 unit (for problem b.):



The sine of a sawtooth?

Connecting the circle definition and the graph of the sine function

Team: Utrechts Stedelijk Gymnasium (USG), The Netherlands

Target knowledge	Discovering the connection between the definition of a (co)sinus in a unit circle and the (co)sinus graph.
Broader goals	Gaining flexibility in conversion between angles in degrees and radians. Experiencing mathematical inquiry. Reasoning. Collaboration.
Prerequisite mathematical knowledge	Trigonometry rules in a right triangle. Angle measures in degrees and radians. Proportions in the standard triangles (45-45-90 and 30-60-90 degree angles) described with square roots. NOT: knowledge of the (co)sinus graph!
Grade	Age 16-17 (end of VWO 4 or start of VWO 5, Mathematics B, in Netherlands)
Time	40-80 minutes
Required material	Clothesline and pegs. Paper sheets with the blank coordinate systems, unit circle, and the sine graph. CDs (or DVDs) with a dot on the edge, one for each group. For each student, a protractor and at least 5 pens in different colours. Digiboard with GeoGebra, and on which one can write (or a whiteboard with markers). Worksheets (for a more guided lesson).

Problem:
You are given a CD with a dot on the edge. Put a finger through the hole in the centre and spin the CD so that the dot moves. Investigate the motion of the dot. Observe the height of the dot while you rotate the CD. Draw a graph showing how the height of the dot depends on the angle of rotation. Simultaneously follow the movement of the dot on the CD and the graph.

Additional problems:
Determine what the values of $\sin 135°$, $\sin 90°$, and $\sin 210°$ are. Explain the connection to the sine in a triangle.
Instead of the vertical movement (the height), consider the horizontal movement of the dot and draw the graph of its dependency on the angle.

Phase	Teacher's actions, including instructions	Students' actions and reactions
Devolution (didactical) 2 minutes	The teacher divides the class into groups of three students. Grade-repeating students sit together in one group, so that any pre-knowledge is not present in many different groups. The teacher shows a CD with a dot on the edge, puts a finger through the hole in the centre and spins the CD so that the dot moves. *"We will investigate the motion of the dot."* The teacher presents the problem to the students, with the instruction to first work with a pencil and to use a pen or a marker only for the final version. Students will hang their results on the clothesline and the teacher explains that it is fine if they hang more than one attempt.	Students listen and consider the materials they are given.

Action and formulation (adidactical/ didactical) 20 minutes	The teacher observes students working and encourages the students to hang their initial drawings on the clothesline. If the teacher wishes to lead the students, she will instruct them to take the centre of the CD as the origin, the radius of the CD to be 1, to consider the angle of rotation in radians, to start with the dot at the far right and to rotate the CD counterclockwise. *A variant of the more guided worksheet is also provided in the additional materials.* After the instruction, the teacher asks the students again to share their graphs.	Students try to draw a graph and explore different possibilities (sawtooth, parabola, circle, etc). Their possible strategies are described below. Students walk along the clothesline and see different graphs. That may influence their reasoning and they may change their drawings. With the additional instruction, the drawings will probably be more similar to each other. Hopefully, the students will start thinking about the right solution.
Formulation (adidactical) 5 minutes	The teacher asks the students to hang their drawings on a clothesline. If the instruction was given, the new drawings are hanged next to the old ones. The teacher ignores the variety of solutions including the apparent misconceptions, leaving this discussion to the students.	Each group hangs a new drawing on the clothesline, next to the previous one. Students walk along the clothesline and see different graphs.
Validation (didactical) 8 minutes	The teacher leads the discussion about the drawings that stood out and the questions they raised. The teacher directs the discussion to the starting point of the graph, angles, radians, and all other characteristics of the graph. To support the students' reasoning the teacher invites the students to follow the dot both on the circle and on the graph. The teacher may present the animation in GeoGebra, in which the point spins over the circle and in the meantime, the students see the height of the point in the graph.	Students engage in the discussion about the different approaches to drawing the graph and express their opinion on which graph is better or the best and why. It is expected that the students agree that the correct solution is the sine graph and that they make the connection between the angle of rotation in the first quadrant and the height of the dot as the sine of the angle.
Institutionalisation (didactical) 5 minutes	When the correct graph form is agreed upon, the teacher hands out the printed version on paper from Appendix 2. (S)he tells the students that this graph is called a sine graph and that it means wave in Latin (or asks the students what it means). Ask whether some students already know about the term sine or have they seen something similar elsewhere.	Students try to remember seeing such graphs in physics, hearing about the sine function in a right triangle, etc.
Devolution (didactical) 2 minutes	The teacher asks the students to use both the unit circle and the graph to answer what the values of $\sin 135°$, $\sin 90°$, and $\sin 210°$ are.	Students consider the printouts with the unit circle and the sine graph.

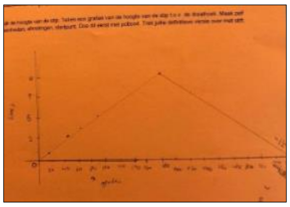
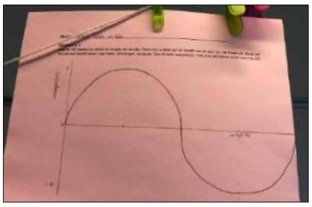
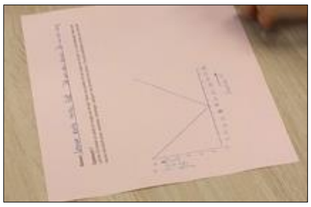
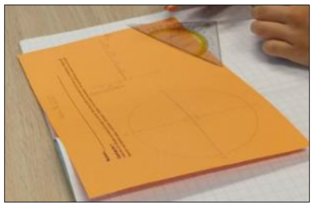
Action (adidactical) 8 minutes	The teacher observes what the students are doing and writing.	Students try to connect the graph with the dot on the circle. They observe the given angles on both models and measure or use other methods to determine the height.
Formulation (adidactical) 4 minutes	The teacher asks the different groups to tell their results and to explain their reasoning.	Students report their findings and explain how they found the results.
Validation (didactical / adidactical) 4 minutes	The teacher asks the students to repeat the reasoning of the groups that have made the connection between the sine in the triangle, in the circle, and in the drawn graph.	Students once more elaborate on the definition of the sine for the angles in the right-angled triangle and the connection to larger angles based on the circle and the graph.
Institutionalisation (didactical) 7 minutes	The teacher summarizes the connection between the sine in the triangle, in the circle, and in the drawn graph.	Students listen and take notes.
Devolution (didactical) 2 minutes	The teacher poses another additional task to consider the dependency of the horizontal movement of the dot on the angle of rotation.	Students listen and consider the CD again.
Action (adidactical) 5 minutes	The teacher observes the students' work. (S)he may consider telling the students that the displacement is considered to be zero in the circle centre (and the vertical line through it).	Students explore, discuss the difference between the previous exercises, and draw a graph.
Formulation (adidactical) 2 minutes	The teacher asks the students to hang their findings on the clothesline.	Each group hangs a drawing on the clothesline, next to the other two. Students walk along the clothesline and look at each other's results.
Validation (didactical/ adidactical) 3 minutes	The teacher discusses with students what stood out and what questions were raised.	Students will point out the "shift" in the graph. They may mention the symmetry coming from turning the CD for 90 degrees.
Institutionalisation (didactical) 3 minutes	The teacher says it has the same shape as the sine graph but is shifted for $\pi/2$ (90 degrees) and that the function is called the cosine.	Students listen and take notes.

Possible ways for students to realize target knowledge

Without any instructions, the students must make their own choices for the axes, units, dimensions, and the starting point. This will raise some questions and cause confusion. They might argue with each other:

- Should the graph be a sawtooth, parabolas glued together, semicircles (the outline along the disk), or something else?
- Do you start with the dot at the top or somewhere else?
- Does the graph stay above the axis or pass through it?
- Should we use degrees or radians?

Students will draw a lot of different graphs. Some will draw a “sawtooth” with straight lines. Others will draw a curvy graph, some will draw half circles etc., all starting at different points. Students will measure the radius of the CD and most of them will use angles in degrees.

After seeing the graphs of other groups, the students will talk in their groups about the different forms of graphs. Maybe they will change their initial thoughts about the form of the graph and draw a different one. If the instruction about all of the above questions is given, the graphs of all groups will probably be more similar. If all goes well, there will be waves that look more or less the same.

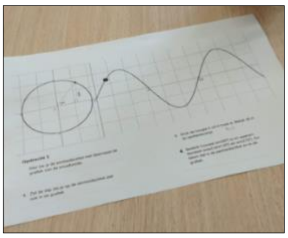
In the end, probably the students will talk about the differences between the first and the second graph they have drawn. They will discuss the right form and hopefully, they will conclude that it must be a wave and not a sawtooth, but probably there will be some groups stubbornly sticking to certain graphs: semicircles, parabolas, or sawtooths.

Some students might try to introduce the (vertical) speed of the dot into the discussion. This should be discussed, and it should be clarified that we are looking at the dependency on the angle, not on time. If the misconception persists, the teacher might also say that the speed of rotation is constant.

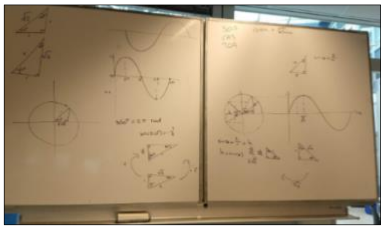
When asked about previous encounters of the sine:

- Students will talk about trigonometry rules in a right triangle.
- They will (more or less) understand why it must be a wave.
- Some of them will refer to the sine graph and the “waves” they have seen in their physics lessons.

To calculate the values of $\sin 135°$, $\sin 90°$, and $\sin 210°$, some students are going to measure the height on the graph or the disk. For some, the first quadrant is easier, because they understand the sine function in the right triangle and then they might be stuck, or they might start thinking about the connections between different quadrants. We expect that the students will eventually come to the conclusions $\sin 135° = \sin 45°$, $\sin 90° = 1$ and $\sin 210° = -\sin 30°$. The values of $\sin 45°$ and $\sin 30°$ will probably be familiar to students as part of their pre-knowledge.


Some students have the feeling that they see the connection but cannot express it in words. They have difficulties formulating their findings. Some students see the connection immediately. The students should put the marks for α and 1 in the circle themselves, but after that, it is a standard case of the sine in a rectangular triangle. This is not routine work: This part of the lesson is essential to integrate the different perspectives of the sine: triangles, circles, and graphs.



In the additional task to consider the horizontal movement of the dot, most students will draw some kind of a wave, but are not really sure what the difference is between this and the previous question. Some students will draw the right graph. And maybe there is still a group that draws a "sawtooth". Most students will agree about the form (wave), but the difficulty will be the starting point and the origin. The relevance of the shift for 90° will probably be considered, but maybe without proper justification. The reasoning for employing symmetry (reversing the vertical and horizontal) might be something that the students can visualize or intuitively think of, but it cannot be expected that this kind of subtle arguments will be given by the students themselves.

The use of radians in the exercise can be difficult for the students, and it might distract the students too much from the main considerations. So, the teacher might decide to only use degrees.

The discussion based on the model of the CD can be continued, e.g., to illustrate solutions to equations like $\sin(x) = \sin(\pi - x)$ or to illustrate the phase differences (with two dots on the CD).

NOTE: See the project's webpage for the following editable materials in Word.

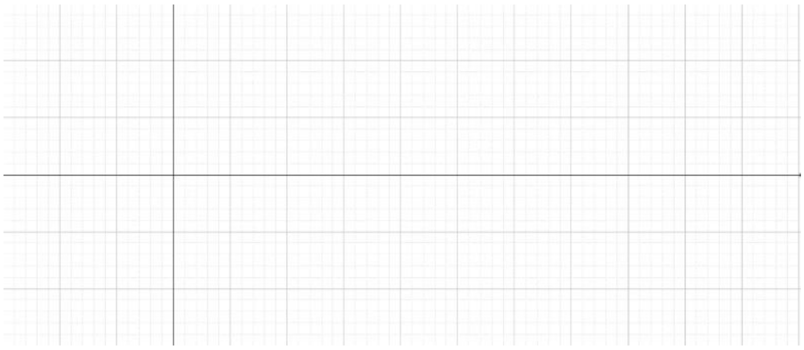
Structured worksheet (give one task at a time!)

TASK 1

Let the CD spin and observe the height of the dot. Draw a graph of the height of the dot relative to the angle of rotation. Make your own choices regarding axes, units, dimensions, and starting point. Do this first with a pencil. Trace your final version with a marker.

TASK 2

Spin the CD and observe the height of the dot. Draw a graph of the height of the dot relative to the angle of rotation. Take the centre of the CD as the origin, the radius of the CD is 1, start with the dot at the far right and rotate the CD exactly one round counterclockwise.



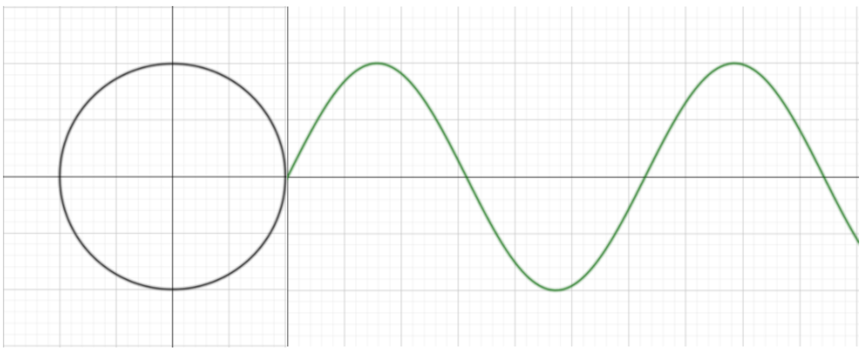
TASK 3

You are given a graph of the unit circle. Also, keep the graph of the sine function.

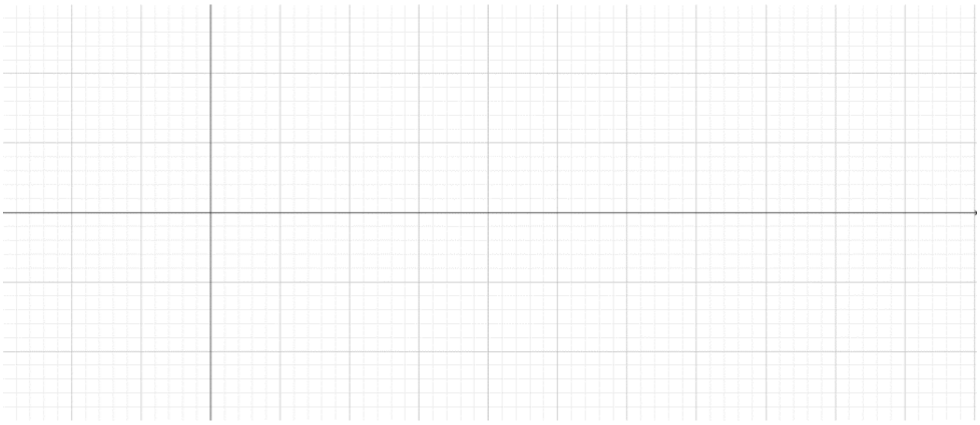
a. Put the dot you see on the unit circle also in the graph.

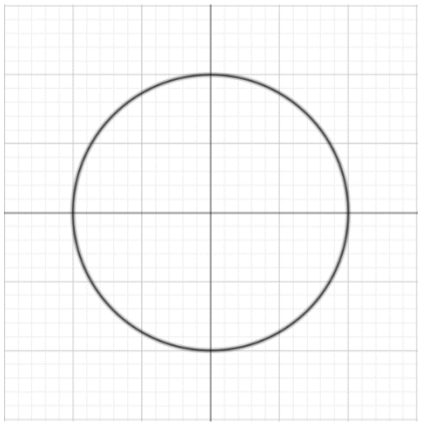
b. What is the connection between the angle α and the height h? Observe this both in the unit circle and in the graph.

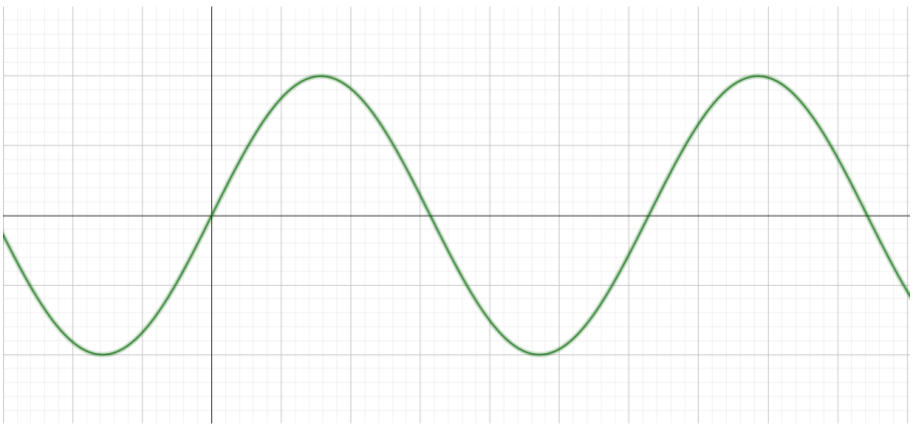
c. What are the values of $\sin(135°)$, $\sin(90°)$ and $\sin(210°)$? In the unit circle and in the graph, mark the relevant points and explain their relevance.



THE BLANK COORDINATE SYSTEM, THE UNIT CIRCLE AND THE SINE GRAPH







Teachers' Inquiry in
Mathematics Education

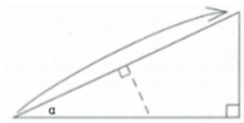
Unfolding Sin(2x)

Trigonometric angle-doubling formulas by folding papers

Team: Utrechts Stedelijk Gymnasium (USG), The Netherlands

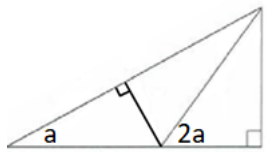
Target knowledge	Discovering the double-angle formulas in trigonometry: $\sin(2x) = 2\sin x \cos x,$ $\cos(2x) = 2\cos^2 x - 1,$ $\cos(2x) = \cos^2 x - \sin^2 x.$
Broader goals	Experiencing mathematical inquiry. Finding a mathematical proof. Reasoning.
Prerequisite mathematical knowledge	Trigonometry rules in a right triangle and the unit circle. Basic acquaintance with trigonometric functions. Simple trigonometric equations.
Grade	Age 16-17 (VWO 5, Mathematics B, in the Netherlands)
Time	60 minutes
Required material	Right-angled triangles (not isosceles, not 30-60-90) from coloured paper, one for each student, different ones for each member of the group. Worksheet. Pencil and (graphic) calculator.

Problem[1]**:**
Find an expression for $\sin(2x)$ and for $\cos(2x)$, using $\sin x$ and $\cos x$.
Use the assignment from the Worksheet.



Phase	Teacher's actions, including instructions	Students' actions and reactions
Devolution (didactical) 10 minutes	The teacher writes the following question on the blackboard: "Is $f(2x) = 2f(x)$ always true?" If necessary, the teacher shows some useful (counter)examples: $f(x) = ax$, $f(x) = \sqrt{x}$, $f(x) = x^2$, $f(x) = e^x$, $f(x) = \ln(x)$. The teacher directs focus on the function $f(x) = \sin x$ and asks: "Is $\sin(2x) = 2\sin x$ always true?" The teacher asks different groups about their conclusion and poses the problem: "What is the right formula for $\sin(2x)$ using $\sin x$? That's what we are going to discover." If necessary, the teacher invites students to recall what they already know about trigonometry in a right-angled triangle.	Students are coming up with some functions. Little discussion in class: the expected answer is "No, not always". Students find other expressions like: $\ln(2x) = \ln(2) + \ln(x)$. Students are trying some values of x, using their calculators. Some are plotting $\sin(2x)$ and $2\sin x$. Students are expected to conclude: "No, but it is true for some values of x". Students recall trigonometry rules in a right triangle and that in a triangle with hypotenuse 1 the length of one side is equal to sin and the length of the other side is equal to cosine.

[1] Source: van Hoeve & van Wijk, uit: (Hull, 2012): Hull, T. (2012). Project Origami: activities for exploring mathematics. In Project Origami (Second Edi). CRC Press.

	The teacher hands out paper triangles and the worksheets with the final task. Make sure that each student in the same group gets a different triangle.	Students consider the worksheets and ask clarifying questions.
Action (adidactical) 30 minutes	The teacher observes the students' work. The Worksheet has an introductory part to find the double angle based on the folding of the paper and the final task in which the students must discover the formula based on the picture. If most students are stuck in the introductory part, the teacher may choose two groups with different approaches, of which at least one group has the correct answer. If the students need additional help with the final task, the teacher may give them hints (in the form of cards or instructions): "*Using the angle, express the lengths of all the sides that are in the picture. Look at the small and big right-angled triangles.*"	Students are busy with exercises 1 to 5 from the Worksheet: folding and finding the desired angles. The first step is to fold and explore angles. Some students will not know the meaning of the phrase "express in a ", so they will only do concrete measurements of the angles. Others might find the folding very difficult. From different examples, they should conclude that the double angle is in the same position in each triangle.  They are going to write down lengths for the sides themselves and based on these, the formulas for $\sin(2\alpha)$ and $\cos(2\alpha)$.
Formulation (didactical/ adidactical) 8 minutes	The teacher lets the students also explain why folding was necessary: "What did you discover by folding?" The teacher asks the students of each group to tell the length of one of the sides of the triangle and writes it on the board. When all lengths are entered, the teacher asks different groups for their formula/conclusion, ending with a group with the correct answer.	Students explain how they found the required angles and that sides which are folded on top of each other are of equal length. Students will tell the length of each side of the triangle. When all lengths are entered, expressions for $\sin(2\alpha)$ and $\cos(2\alpha)$ are formulated, based on the lengths of the sides of the triangle, expressed in $\sin\alpha$ and $\cos\alpha$. Students of the group with the right answer explain their reasoning.
Validation (didactical/ adidactical) 10 minutes	The teacher leads the discussion: "Did it work? Have we found a way to express $\sin(2\alpha)$ in $\sin\alpha$? When are you satisfied?" The discussion should enable all students to ask questions, particularly those that did not come to the formulas on their own.	Students ask any questions that come to their minds and make new insights for themselves based on the formulation of other students. Some students will not be able to see if they have succeeded. "Why can it also contain a $\cos\alpha$?"

Institutionalisation (didactical/ adidactical) 2 minutes	The teacher makes sure that the correct reasoning and formulas are written on the blackboard. Shows where the students can find this in their books.	If necessary, students look up these formulas in the book, so that everyone knows where they are and has seen them.

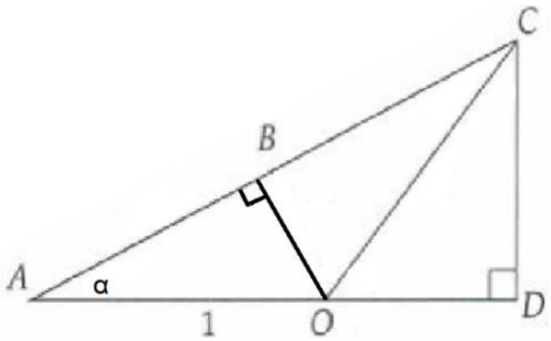
Possible ways for students to realize target knowledge

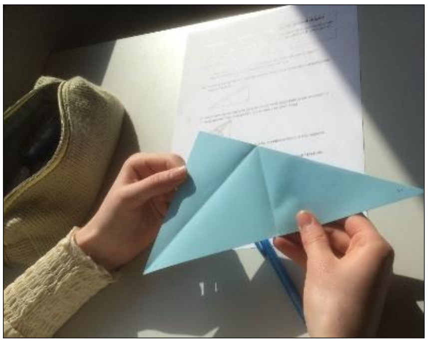
The question to "find a formula for $\sin(2x)$" might be very difficult for the students. What is the desired result? When are we satisfied?

Some students might find the right answer and yet not experience a feeling of success, and some students might be very happy with something we do not like or expect. For example, why is $\sin(2\alpha) = \cos(\pi - 2\alpha)$ not a satisfying answer, but $\sin(2\alpha) = 2\sin\alpha\cos\alpha$ is? Why are we happy with the formula $\cos(2\alpha) = \cos^2\alpha - \sin^2\alpha$? Some students do not think that is any more useful than $\sin(2\alpha)$. Can this be solved by emphasizing in the introduction that you express $f(2x)$ in terms of $f(x)$?

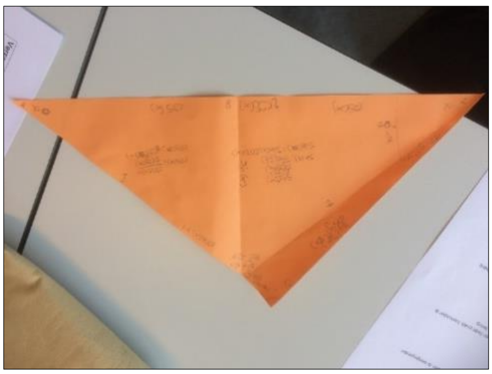
Because of all these reasons, this lesson is planned in the manner of *guided reinvention*, so that the lesson has more structure and students are guided more than in a usual inquiry-based lesson.

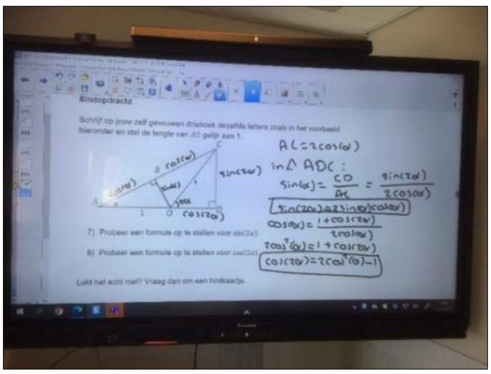
After obtaining the picture below, the students need to convince themselves that the triangles ABO and CBO are congruent. This follows from the fact that by folding the triangle ABO we obtain exactly the triangle CBO. From this, it follows that the angles BAO and BCO are the same and that the angle COD is double their size.





After this phase, the students are directed to express the lengths of all visible segments using the trigonometric functions of the given angle. The students might do this on the folding model, or the worksheet.





We have the following expressions following from the smaller triangles ABO, CBO and CDO:

$$|AO| = |CO| = 1, \qquad |BO| = \sin\alpha$$
$$|AB| = |CB| = \cos\alpha, \qquad |AC| = 2\cos\alpha$$
$$|OD| = \cos(2\alpha), \qquad |CD| = \sin(2\alpha)$$

From the starting triangle ADC we have:

$$\sin(2\alpha) = |CD| = |AC|\sin\alpha,$$

so, by substituting $|AC| = 2\cos\alpha$ we get the formula

$$\sin(2\alpha) = 2\sin\alpha\cos\alpha.$$

The formula for the cosine can be obtained similarly by noting that

$$1 + \cos(2\alpha) = |AD| = |AC|\cos\alpha$$

and by substituting $|AC| = 2\cos\alpha$, we obtain

$$\cos(2\alpha) = 2\cos^2\alpha - 1.$$

If we use the fact, following Pythagoras' theorem, that $\sin^2\alpha + \cos^2\alpha = 1$, we obtain a more familiar formula

$$\cos(2\alpha) = \cos^2\alpha - \sin^2\alpha.$$

Other ways are also possible. To give one more, let us apply Pythagoras' theorem to the triangle ADC:

$$|AD|^2 + |CD|^2 = |AC|^2$$

$$(1 + \cos(2\alpha))^2 + \sin^2(2\alpha) = (2\cos\alpha)^2$$

$$1 + 2\cos(2\alpha) + \cos^2(2\alpha) + \sin^2(2\alpha) = 4\cos^2\alpha$$

Again, by using that $\cos^2(2\alpha) + \sin^2(2\alpha) = 1$ and dividing by 2 we obtain:

$$\cos(2\alpha) = 2\cos^2\alpha - 1.$$

NOTE: See the project's webpage for the following editable materials in Word.

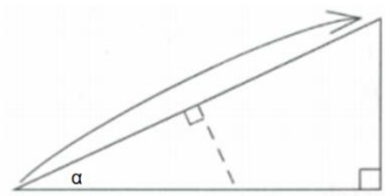
WORKSHEET

In this lesson, we are going to explore some of the rules for trigonometric ratios and we are going to do so by folding a paper triangle.

You have in front of you a rectangular triangle made of coloured paper.

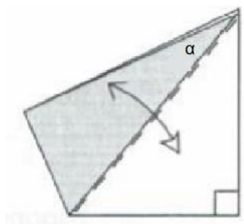
Name the smallest angle α.

Fold the small corner α to the other corner as shown below with the dotted line.



Then fold along the side you just obtained as shown below with the dotted line.

Then unfold everything back.



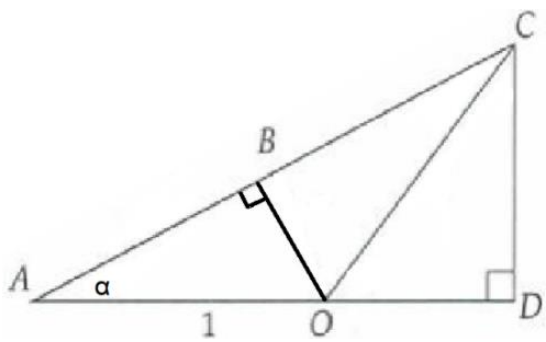
If all goes well, you will see that your triangle is now divided into three smaller triangles.

What other angles can you find in your triangle? Express those angles in α.

Where do you see the double of your angle α, i.e., angle 2α?

The final task

On your folded triangle, write the letters as in the example below and set the length of AO equal to 1.



Using this picture, try to set up formulas for $\sin(2\alpha)$ and $\cos(2\alpha)$.

Apricot jam

The graph of a function and what can be observed from it

Team: Gimnazija Franc Miklošič Ljutomer, Slovenia

Target knowledge	Using the basic characteristics of functions (the increasing/decreasing interval, the sign, the zero, the initial value, extreme values, etc.) to model situations from everyday life.
Broader goals	Communication. Development of inquiry skills. Data analysis. Connection of mathematics to the real situation.
Prerequisite mathematical knowledge	Linear function and its properties. Modelling with linear functions. The use of data processing and graphing programs.
Grade	Age 16 (2nd grade in Slovenia)
Time	90 minutes
Required material	A document with three tables and a graph. The tables include average daytime temperatures, minimum temperatures at 5:00, and temperatures at 7:00, 14:00, and 21:00. CAS (e.g., GeoGebra, Excel, Graphing calculator) to plot graphs. Reporting tools (e.g., Padlet or Jamboard)

Problem:
During the summer holidays (2020), Jaka went to visit his grandmother, who lives in a village near Murska Sobota. Jaka cries: *"Grandma, I am so hungry. Can I have a piece of bread with your good homemade apricot jam?"* and the grandma answered: *"I'm sorry. I couldn't make jam this year. Spring weather is to blame for that.*" Is grandma right about the weather?

Phase	Teacher's actions, including instructions	Students' actions and reactions
Devolution (didactic) 5 minutes	The teacher presents the problem and invites the students to write down (in a joint document) what they think might be the reason that grandma did not make jam due to the weather (2020). The teacher invites students to consider where to find air temperature data for the time and place mentioned.	Students listen and try to understand the problem.
Action (adidactic) 10 minutes	The teacher observes the students' work (what they are searching for, are they discussing, and writing conclusions) and writes their answers in the joint document. Attention is given to answers based on the air temperature.	Students think about possible reasons and search the web for the addresses of the site where they could find relevant information about the air temperature. They write their opinions (e.g., in a joint document using Jamboard).
Formulation (didactic) 5 minutes	The teacher invites students to share their opinions with the whole class. This provides an opportunity to direct all the students to consider the data about the air temperature.	Students share their findings and reasoning and participate in the debate. It is expected that there will be many different opinions and that only some students will use mathematical arguments.
Devolution (didactic) 5 minutes	The teacher divides the class into groups of 3-4 students. Each group gets three tables with air temperatures and a graph of the regression function that is	Students look at the tables, listen to the instructions, and ask questions if they don't understand what their task is.

	the best fit for the data in one of the tables. The teacher instructs the students to reason using the tables whether there could have been spring frost. In case there was frost, they should find the date of the frost.	Students investigate changes in the air temperature for each table and match the graph of the regression function with the data from the table that they think matches the plotted function graph the most.
Action (adidactic) 25 minutes	The teacher observes the students' work without interference. In preparation for the formulation phase, the teacher marks the strategies of the groups and thinks about the order of presentations.	Students analyse the given tables and the plotted graph within their group. They write the data into a joint document, each group on its own page. It is expected that they write additional data on the given graph and draw new graphs. They write down their arguments and answers to the introductory question.
Formulation (didactic) 5 minutes	The teacher invites some students to present their group's findings. The goal is to hear as many different reasonings as possible based on the properties of functions.	The students that are invited by the teacher present the findings of their group. Students are expected to mention some of the following: - the initial and final temperature value (for the observed period) - the moments in which the temperature was zero - intervals during which the air temperature was negative or positive - the minimum/maximum air temperature of the day - intervals during which the temperature has been increasing or decreasing - intervals during which the temperature was below -2˚C, when the blossoms can freeze - arguments for matching tables and graphs of the regression functions - arguments supporting whether there has been spring frost and a comparison of the dates obtained
Validation (didactic) 15 minutes	The teacher organizes a class discussion in which all arguments are repeated and questioned. The teacher directs the discussion towards a joint conclusion about what actually happened, supporting the students to use mathematical language.	Students discuss the presentations of other groups. Some arguments may be repeated or reformulated if they are not clear to everyone. Following the teacher's cues, the students try to find a common conclusion.
Institutionalisation (didactic) 5 minutes	The teacher makes a summary and connects students' solutions and reasoning to the mathematical terms used for properties of functions (the increasing/decreasing interval, the sign, the zero, the initial value, the extreme value, etc.)	Students listen and add definitions and anything they find was new to them in this lesson to their notes.

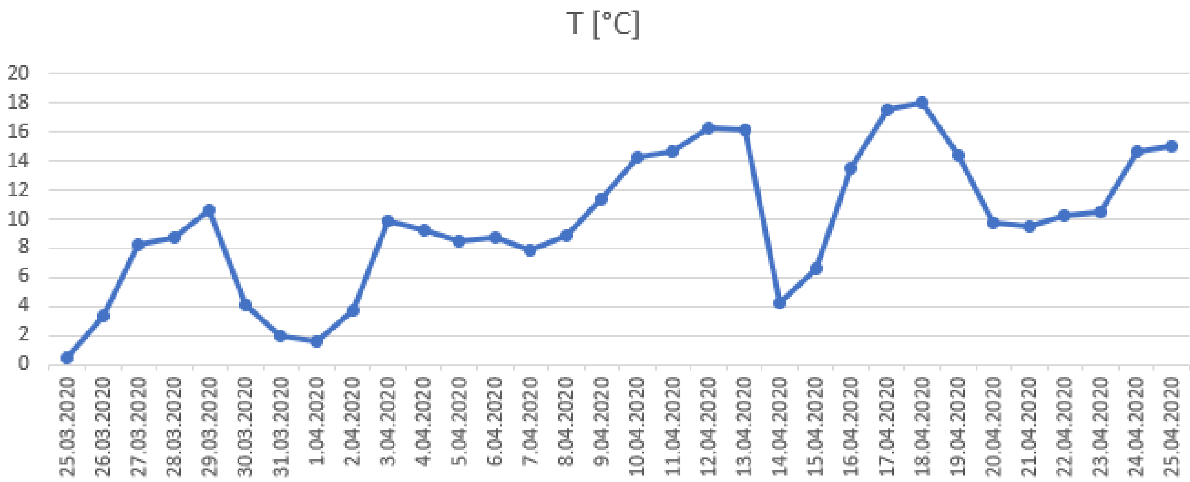
Possible ways for students to reach the target knowledge

The given graph of the regression function corresponds to the data in Table 2. The students should eliminate Table 1 because the temperatures are all positive, while they should eliminate Table 3 because there are too many 'fluctuations', i.e., a lot more alternations of local minima and maxima than in the regression graph.

To analyse the data in the given tables, students may draw discrete graphs and connect the data points to obtain graphs of a piecewise linear function. These graphs may be used to compare the data with the given graph of a regression function. On the other hand, or in addition, some conclusions can be deduced directly from the tables.

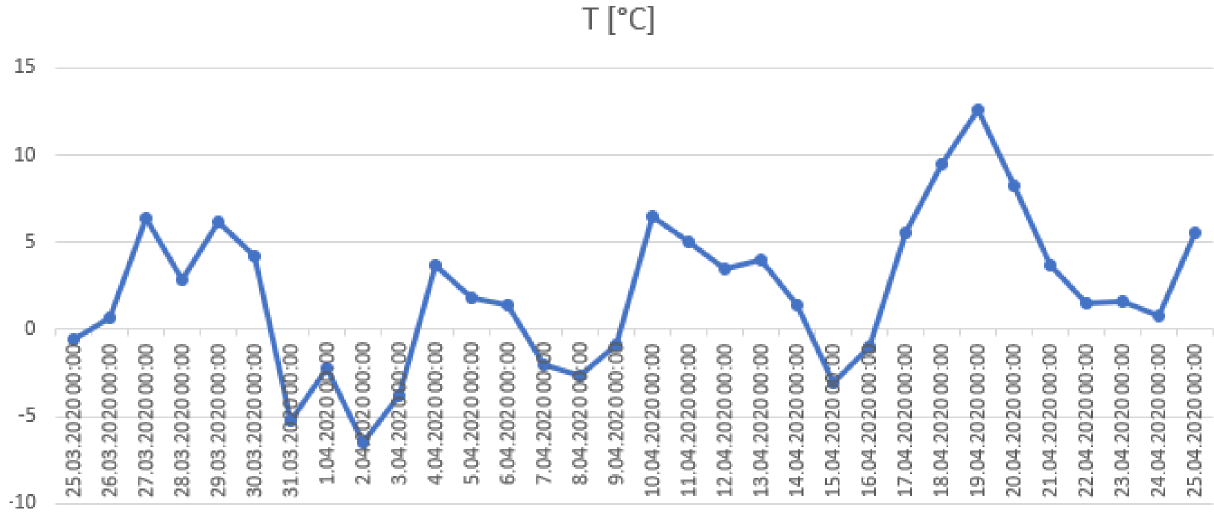
For each table, we give the graph and some properties of the data.

Table 1:



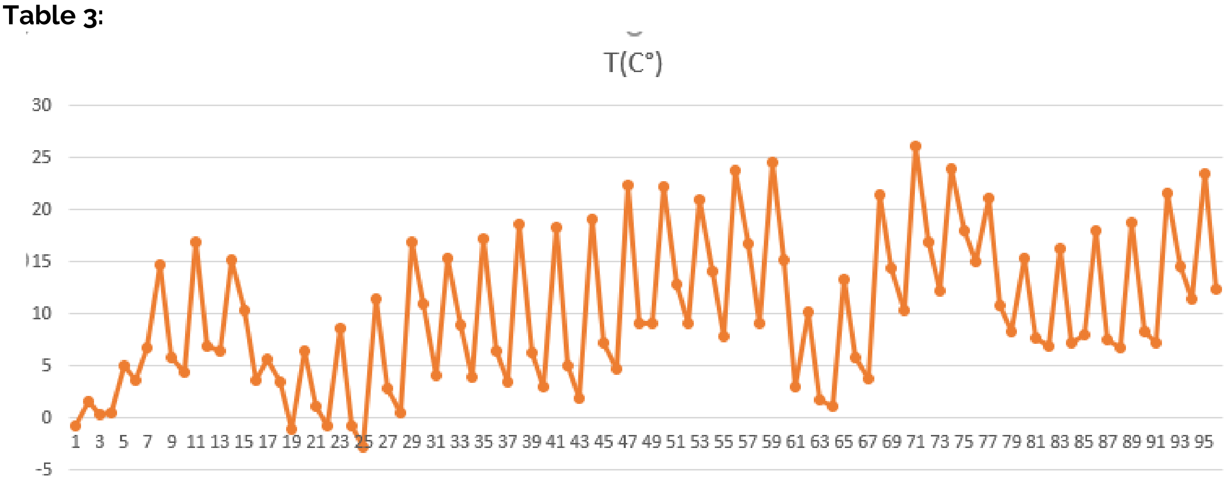
- On 25. 3. 2020, there was the lowest average daily temperature.
- On 18. 4. 2020, there was the highest average daily temperature.
- The average daily temperature was positive every day.
- From 25. 3. until 29. 3., the temperature was rising, then it was falling until 1. 4., then it was rising again until 13. 4., then it was falling again a little and was equal to around 10 °C.

Table 2:



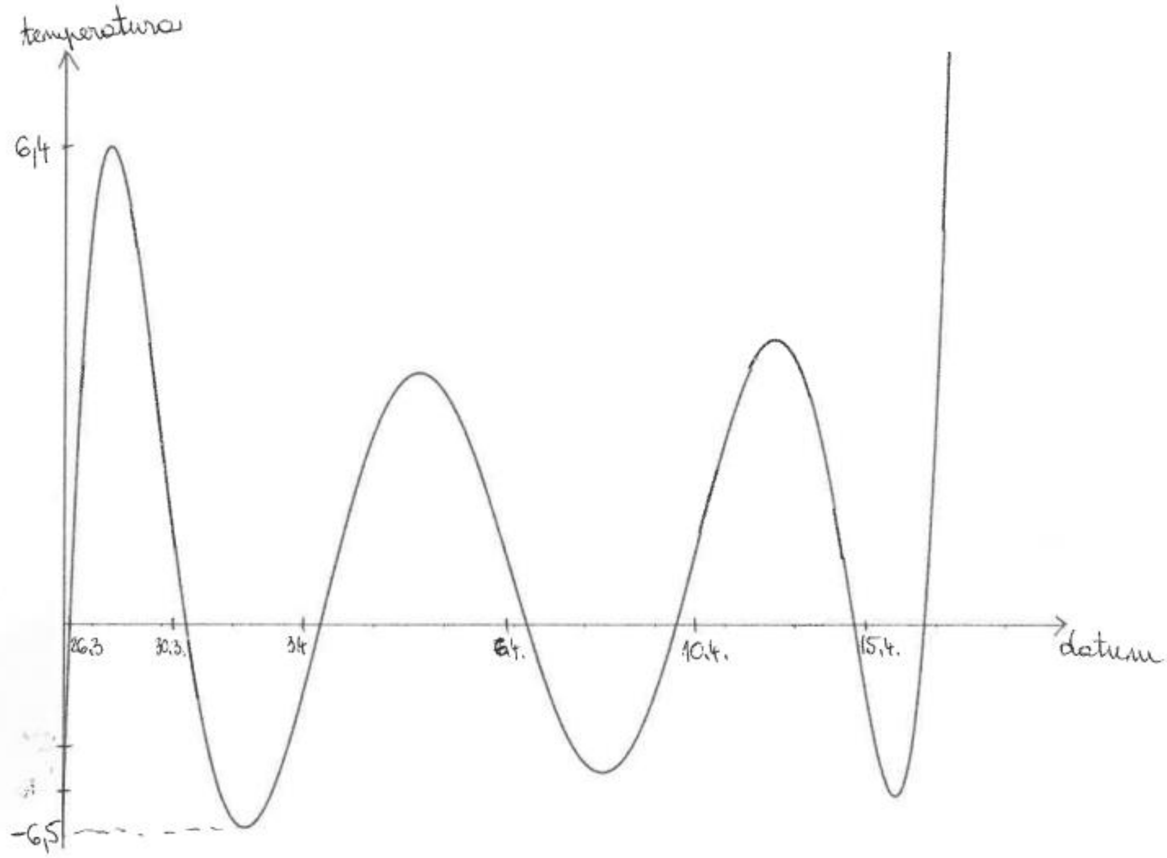
- The lowest minimum daily temperature was on 2. 4. 2020.
- The minimum daily temperature was negative on 25. 3., from 31. 3. to 3. 4., on 8. 4., on 9. 4. and on 15. 4. 2022.
- The minimum daily temperature fluctuated a lot, first it was negative, then it was positive, then negative again, positive …

Table 3:



- The lowest temperature was on 2. 4. 2020 at 7:00, before that the temperatures were quite high.
- The highest temperature was at 14:00 on 17. 4. 2020.
- Low temperatures (around 0 ˚C) were on 25. 3. and 26. 3., 31. 3. - 2. 4., 14. 4., 15. 4.
- On 27. 3. and 28. 3. it was quite warm and after that quite cold.
- After 2. 4. the temperatures were positive.

To answer when the frost occurred, students could mark the coordinate axes with the corresponding date and temperature, and observe that the frost occurred on 2. 4. 2020.



In the given graph, the following observations could be made:

- The temperature value at the beginning of the observation is -0.6˚ C, and at the end 5.5 ˚C.
- The temperature was zero at some point between 25. 3. and 26. 3, 30. 3. and 31. 3., 3. 4. and 4. 4., 6. 4. and 7. 4., 9. 4. and 10. 4., 14. 4. and 15. 4., 16. 4. and 17. 4.
- The air temperature was negative on 25. 3, from 31. 3. to 3. 4., from 7. 4. to 9. 4., from 15. 4. to 16. 4.
- The air temperature was positive on the remaining days when it was not negative.
- The minimum temperature was on 2. 4., and the maximum air temperature was on 19. 4.
- The temperature increased from 25. 3. to 27. 3., from 2. 4. to 4. 4., from 8. 4. to 10. 4., from 15. 4. to 19. 4.
- The temperature has been falling from 29. 3. to 2. 4., from 4. 4. to 8. 4., from 13. 4. to 15. 4., from 19. 4. to 24. 4.
- The temperature had a value below -2˚ C, when the blossoms of fruit trees that just started to open can freeze, from 31. 3. until 3. 4., on 8. 4. and on 15. 4.
- First, the temperature was negative, then it was increasing until 27. 3., then decreased, became negative, then increased and was positive again.
- The minimum temperatures were well above 0˚ C, but then on 2. 4. the temperature dropped significantly and was negative, so it is most likely that the frost on apricots occurred on this date.

In the summary the teacher may formulate these findings mathematically in the following way:

- The zero is where the temperature was equal to 0 °C, that is, the point where the graph intersects the horizontal axis.
- The initial value is the value of the regression function for $x = 0$ (the moment in which we start measurements), the ordinate of the intersection of the graph with the vertical axis.
- The maximum value is the highest temperature in the table or the maximum in the graph, and the minimum value of the function is the lowest temperature in the table or the minimum of the graph.
- The interval on which the function is positive/negative are those days for which temperatures were positive/negative.
- The increasing/decreasing interval are those days when the temperature increased/decreased.
- The domain of the graphed function is the set of all days on which temperatures were recorded, and the image are the different measured temperatures (from the lowest to the highest).

NOTE: See the project's webpage for the following editable materials in Word.

TABLE 1

MURSKA SOBOTA – RAKIČAN (average daytime temperature) lon=16.1913, lat=46.6521, h=187m	
Date	***T [°C]***
2020-03-25	0,4
2020-03-26	3,3
2020-03-27	8,3
2020-03-28	8,8
2020-03-29	10,6
2020-03-30	4,1
2020-03-31	2
2020-04-01	1,6
2020-04-02	3,7
2020-04-03	9,9
2020-04-04	9,3
2020-04-05	8,5
2020-04-06	8,7
2020-04-07	7,9
2020-04-08	8,9
2020-04-09	11,4
2020-04-10	14,3
2020-04-11	14,6
2020-04-12	16,3
2020-04-13	16,1
2020-04-14	4,2
2020-04-15	6,6
2020-04-16	13,5
2020-04-17	17,6
2020-04-18	18,1
2020-04-19	14,4
2020-04-20	9,8
2020-04-21	9,5
2020-04-22	10,3
2020-04-23	10,5
2020-04-24	14,6
2020-04-25	15

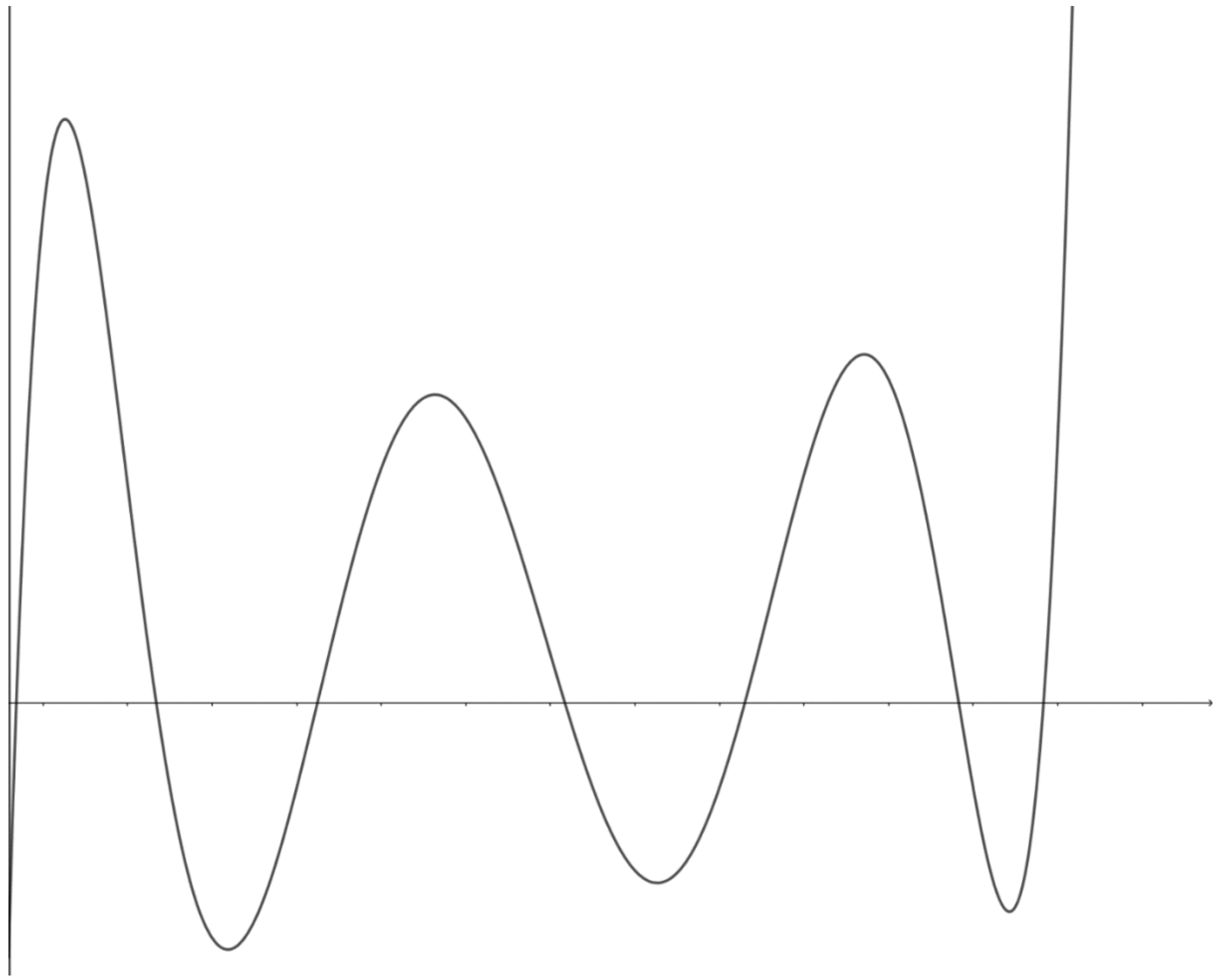
TABLE 2

MURSKA SOBOTA – RAKIČAN (minimum daytime temperature) lon=16.1913, lat=46.6521, h=187m	
Date and time	***T [°C]***
2020-03-25 05:00	-0,6
2020-03-26 05:00	0,7
2020-03-27 05:00	6,4
2020-03-28 05:00	2,9
2020-03-29 05:00	6,2
2020-03-30 05:00	4,2
2020-03-31 05:00	-5,2
2020-04-01 05:00	-2,2
2020-04-02 05:00	-6,5
2020-04-03 05:00	-3,8
2020-04-04 05:00	3,7
2020-04-05 05:00	1,8
2020-04-06 05:00	1,4
2020-04-07 05:00	-2
2020-04-08 05:00	-2,6
2020-04-09 05:00	-0,9
2020-04-10 05:00	6,5
2020-04-11 05:00	5
2020-04-12 05:00	3,5
2020-04-13 05:00	4
2020-04-14 05:00	1,4
2020-04-15 05:00	-3,1
2020-04-16 05:00	-1
2020-04-17 05:00	5,5
2020-04-18 05:00	9,5
2020-04-19 05:00	12,6
2020-04-20 05:00	8,3
2020-04-21 05:00	3,7
2020-04-22 05:00	1,5
2020-04-23 05:00	1,6
2020-04-24 05:00	0,8
2020-04-25 05:00	5,5

TABLE 3

MURSKA SOBOTA – RAKIČAN (the temperature at certain times) lon=16.1913, lat=46.6521, h=187m					
Date and time	***T [°C]***	***Date and time***	***T [°C]***	***Date and time***	***T [°C]***
2020-03-25 07:00	-0.7	*2020-04-04 21:00*	8.9	*2020-04-15 14:00*	13.3
2020-03-25 14:00	1.6	*2020-04-05 07:00*	4	*2020-04-15 21:00*	5.9
2020-03-25 21:00	0.3	*2020-04-05 14:00*	17.2	*2020-04-16 07:00*	3.8
2020-03-26 07:00	0.5	*2020-04-05 21:00*	6.4	*2020-04-16 14:00*	21.5
2020-03-26 14:00	5.1	*2020-04-06 07:00*	3.5	*2020-04-16 21:00*	14.4
2020-03-26 21:00	3.7	*2020-04-06 14:00*	18.6	*2020-04-17 07:00*	10.4
2020-03-27 07:00	6.8	*2020-04-06 21:00*	6.3	*2020-04-17 14:00*	26.1
2020-03-27 14:00	14.8	*2020-04-07 07:00*	3	*2020-04-17 21:00*	17
2020-03-27 21:00	5.8	*2020-04-07 14:00*	18.4	*2020-04-18 07:00*	12.3
2020-03-28 07:00	4.5	*2020-04-07 21:00*	5.1	*2020-04-18 14:00*	24
2020-03-28 14:00	16.9	*2020-04-08 07:00*	2	*2020-04-18 21:00*	18
2020-03-28 21:00	6.9	*2020-04-08 14:00*	19.1	*2020-04-19 07:00*	15
2020-03-29 07:00	6.5	*2020-04-08 21:00*	7.2	*2020-04-19 14:00*	21.1
2020-03-29 14:00	15.2	*2020-04-09 07:00*	4.7	*2020-04-19 21:00*	10.8
2020-03-29 21:00	10.4	*2020-04-09 14:00*	22.4	*2020-04-20 07:00*	8.3
2020-03-30 07:00	3.7	*2020-04-09 21:00*	9.2	*2020-04-20 14:00*	15.4
2020-03-30 14:00	5.7	*2020-04-10 07:00*	9.2	*2020-04-20 21:00*	7.7
2020-03-30 21:00	3.5	*2020-04-10 14:00*	22.3	*2020-04-21 07:00*	6.9
2020-03-31 07:00	-1	*2020-04-10 21:00*	12.8	*2020-04-21 14:00*	16.3
2020-03-31 14:00	6.4	*2020-04-11 07:00*	9.1	*2020-04-21 21:00*	7.3
2020-03-31 21:00	1.2	*2020-04-11 14:00*	21	*2020-04-22 07:00*	8.1
2020-04-01 07:00	-0.8	*2020-04-11 21:00*	14.2	*2020-04-22 14:00*	18
2020-04-01 14:00	8.6	*2020-04-12 07:00*	7.8	*2020-04-22 21:00*	7.5
2020-04-01 21:00	-0.8	*2020-04-12 14:00*	23.8	*2020-04-23 07:00*	6.7
2020-04-02 07:00	-2.7	*2020-04-12 21:00*	16.8	*2020-04-23 14:00*	18.8
2020-04-02 14:00	11.5	*2020-04-13 07:00*	9.2	*2020-04-23 21:00*	8.3
2020-04-02 21:00	2.9	*2020-04-13 14:00*	24.6	*2020-04-24 07:00*	7.3
2020-04-03 07:00	0.5	*2020-04-13 21:00*	15.2	*2020-04-24 14:00*	21.7
2020-04-03 14:00	16.9	*2020-04-14 07:00*	3	*2020-04-24 21:00*	14.6
2020-04-03 21:00	11	*2020-04-14 14:00*	10.2	*2020-04-25 07:00*	11.5
2020-04-04 07:00	4.1	*2020-04-14 21:00*	1.7	*2020-04-25 14:00*	23.5
2020-04-04 14:00	15.4	*2020-04-15 07:00*	1.2	*2020-04-25 21:00*	12.4

GRAPH OF THE REGRESSION FUNCTION



Teachers' Inquiry in
Mathematics Education

The attic room

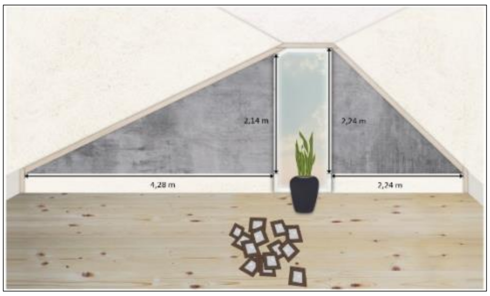
Investigating geometric patterns to calculate sums

Team: Gimnazija Franc Miklošič Ljutomer, Slovenia

Target knowledge	Discovering the rules for the sum of the first n positive integers and the sum of the first n odd positive integers.
Broader goals	Development of inquiry skills. Investigation of patterns. Discovery of connections between geometry, numbers, and algebra. Development of mathematical thinking. Use of mathematical language and symbols. Presentation and interpretation of the results of own inquiry.
Prerequisite mathematical knowledge	Basic arithmetic and properties of addition of natural numbers.
Grade	Age 15-16 (1st grade in Slovenia)
Time	90 minutes
Required material	Worksheet. Pen and paper.

Problem:

Hansel and Gretel are arranging the attic room. On the grey parts of the triangular shape of one wall (see the picture), they will hang framed photos with dimensions of 20 cm x 20 cm without overlapping.
Explore the wall tiling with the maximal number of photos without overlap.



Phase	Teacher's actions, including instructions	Students' actions and reactions
Devolution (didactical) 5 minutes	The teacher presents the setting of the two walls and photos. (S)he invites the students to pose many different questions about this setting.	Students listen and consider the handout.
Formulation (didactical) 5 minutes	The teacher listens to the students' suggestions and summarizes the main aspects of the problem.	Possible students' questions: *What is the hypotenuse of triangles? What are the sizes of the angles of triangles? What is the wall area that is left empty (triangles)? What part of the wall will be left empty? What is the wall area tiled by the photos?* *How many photos fit on each wall?* *How many more photos are on the larger, compared to the smaller wall?*
Devolution (didactical) 5 minutes	After a short discussion, the teacher divides the students into working pairs and directs their focus to the main question of finding the maximum number of photos on each wall. (S)he invites them to write down a general formula that would work with different dimensions of photos. Students should keep in mind that framed photos cannot be cut.	Students may ask additional questions before proceeding to the joint task.

Action (adidactical) 40 minutes	The teacher observes the work of the students. Students might focus only on one wall and have different strategies. The teacher prepares for the order in which the formulation will take place.	Students research and write down/draw their findings. It is expected that they will draw various pictures and start with concrete calculations. Some students might look at smaller examples.
Formulation (didactical) 15 minutes	The teacher organizes the presentation of students' solutions.	Students report their findings. They explain how they came up with their solution.
Validation (didactical) 10 minutes	The teacher invites the students to compare different approaches and results in an open discussion. Different formulas should be checked with concrete examples. Geometrical and algebraical reasoning should be linked.	Students may notice many similarities and differences among their solutions. They respond to teachers' questions or guide the discussion based on their considerations. If they have not reached the formulas themselves, the students ask clarifying questions.
Institutionalisation (didactical) 10 minutes	The teacher makes a summary of the students' solutions and emphasizes the two formulas.	Students write down the conclusions and formulas in their notebooks.

Possible ways for students to realize target knowledge

The first consideration could be done by calculating the area of a triangle and dividing that area by the area of one photo. This leads to an upper bound on the number of photos, but this bound cannot be reached because the wall is triangular and there will be parts of the wall not covered by photos.

For the left wall:

$$\frac{(428\ cm \cdot 214\ cm)}{2} : (20\ cm \cdot 20\ cm) = 45796 : 400 = 114{,}96.$$

For the right wall:

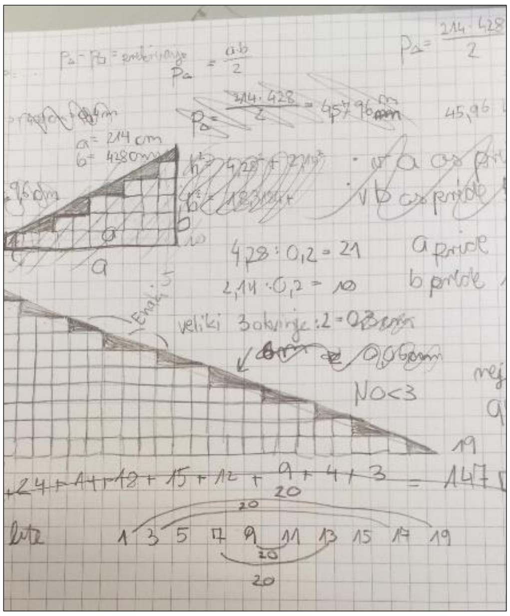
$$\frac{(224\ cm \cdot 224\ cm)}{2} : (20\ cm \cdot 20\ cm) = 25088 : 400 = 62{,}72.$$

Some students might claim that the answers are 114 and 62 photos, but this has to be challenged by the need for an explicit arrangement that achieves so many photos.

Students who draw a sketch of the wall to scale might find that on the left wall, in each subsequent row, the number of squares increased by two and on the right by one.

For the left wall: $1 + 3 + 5 + 7 + 9 + 11 + 13 + 15 + 17 + 19 = 100.$

For the right wall: $1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 + 10 = 55.$



In this picture, we see that a student drew the complete wall with maximal tiling. Note however that the picture also shows that in some rows the number of photos increases by two, while in others it increases by three. In the end, the correct sum has been written and the calculation has been done by pairing the numbers.

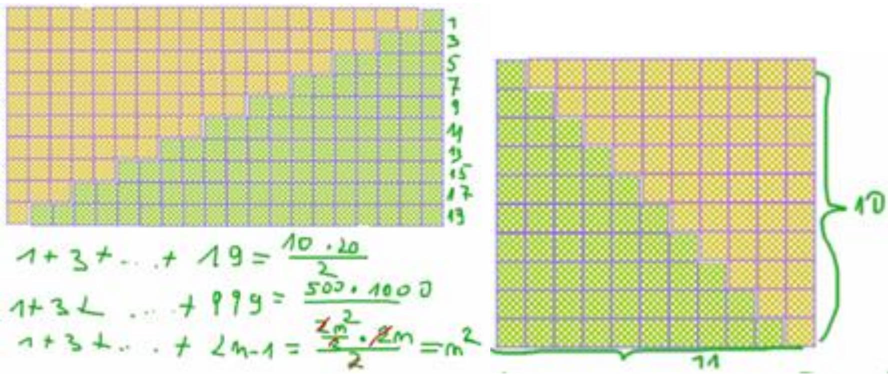
This strategy can be successfully generalized. After observing what the correct sum is, the students may use the idea of pairing to calculate it also with general n.

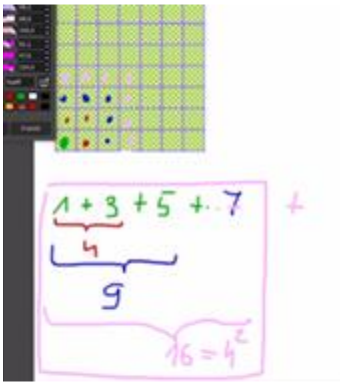
For the right wall a similar strategy would yield:

$$(1+10)+(2+9)+(3+8)+(4+7)+(5+6)$$
$$=5\cdot 11=\frac{10}{2}\cdot 11=\frac{10\cdot 11}{2}.$$

From this, students may hypothesize that for n lines the sum is equal to $\frac{n(n+1)}{2}$.

Some students might consider completing the triangular wall into a rectangular wall. The calculations behind this approach are very similar, only now we have a visualization of the pairing.





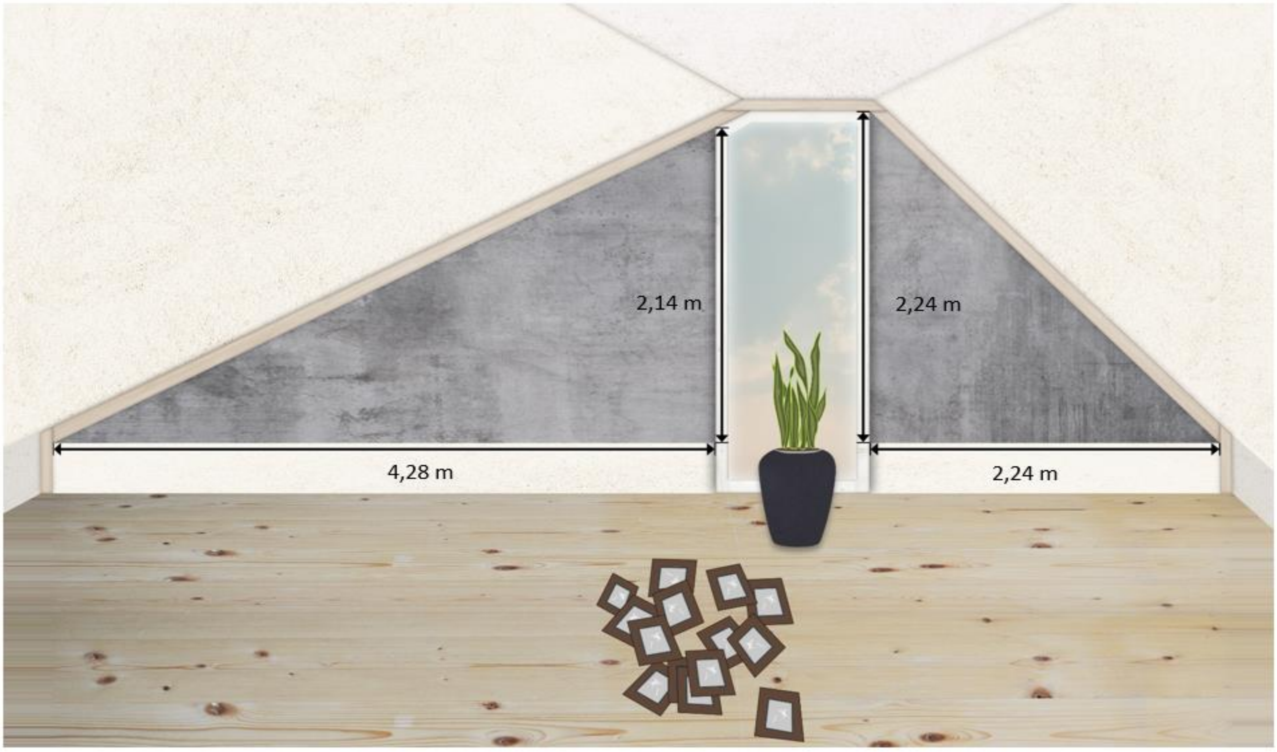
For the calculation of the sum of odd numbers, another strategy is to visualize odd numbers as L-shapes that complete a to a square.

In the end, the students should reach the following formulas.
The sum of the first n odd positive integers: $1+3+\cdots+(2n-1)=n^2$
The sum of the first n positive integers: $1+2+3+\cdots+n=\frac{n(n+1)}{2}$

NOTE: See the project's webpage for the following editable materials in Word.

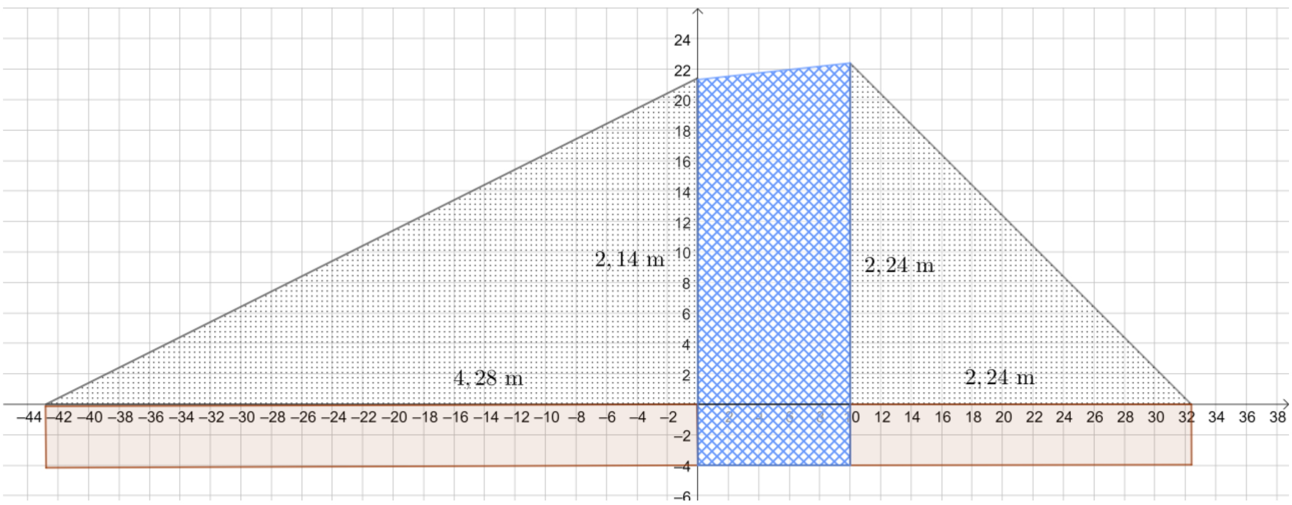
HANDOUT

Hansel and Gretel are arranging the attic room. On the grey parts of the triangular shape of one wall (see picture) they will hang framed photos with dimensions of 20 cm x 20 cm without overlapping. Explore the wall tiling with the maximal number of photos without overlap.



For the teacher

Layout grid:



The birthday paradox

Mathematical explanation of counterintuitive probability facts

Team: Gimnazija Jesenice, Jesenice, Slovenia

Target knowledge	Calculating probability and determining the number of people in the model for the probability to be higher than 50%.
Broader goals	Improving the use of basic laws of probability. Understanding the asymptotic behaviour – concretely, of the function $$f(n) = 1 - \frac{365!}{(365-n)!\,365^n}$$ Problem-solving and modelling skills: Organizing and analysing data. Interpreting the result of a calculation. Acknowledging the application of mathematics in everyday situations. Value of mathematics as a tool that surpasses our intuition. Understanding a veridical paradox.
Prerequisite mathematical knowledge	A-priori definition of probability, the product rule and the probability of the complement.
Grade	Age 17-18 (4th grade in Slovenia)
Time	60 minutes
Required material	Handouts, small papers to write down birthday dates, bag, calculators.

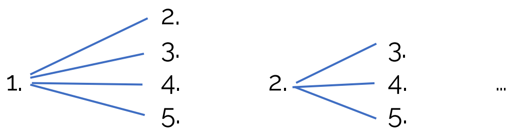
Problem:
How many people do you think it would take on average, to find two people who share the same birthday? How many people do you need in the room to have a 50% chance?

Phase	Teacher's actions, including instructions	Students' actions and reactions
Devolution (didactical) 10 minutes	The teacher invites the students to write down their birthday (date, not the year) and the birthday of five people they know outside of the class. All papers are put in a joint bag. The teacher invites a volunteer to randomly pick five papers out of the bag. The teacher asks the students to share their opinions on how many papers should be chosen *to make sure* we get a pair of the same birthdays. The teacher invites a second volunteer to randomly pick 20 papers out of the bag. If there is no pair of the same birthdays, the volunteer keeps on picking more papers from the bag. When a match appears, the teacher asks the students if they are surprised. The teacher introduces the problem and discusses with the students the assumptions that will be made in the problem: no twins in the group, regular (not leap) year, all days are equally probable, etc.	Students get engaged by the activity that seems mysterious at first. Students notice that there is no pair of the same birthdays on the chosen papers (probably). Students share their answers and explain their reasoning. It is expected that the most common answer will be 366. Numbers could also be written on the board. Students are excited because there is anticipation, and the situation is contrary to our intuition – at least some students would expect a much higher number of papers needed for a match. Students listen, and possibly ask clarifying questions to make sure they understand their task.

Action (adidactical) 20 minutes	The teacher observes the work of the students without interfering. It is expected that some groups will need help because it is much easier to calculate the probability of the complement, while the students might not think of that. In case the students are stuck, the teacher might suggest to that group to consider calculating the probability of not having a pair of the same birthdays among five papers and then generalize for an arbitrary number of people. The amount of guidance should be reduced to a minimum and chosen according to the broader goals that the teacher wants to achieve. Although technology is allowed, students should not search for the solution to the "birthday paradox" on the internet as that would ruin the potential of the task.	Students work in groups. If needed, students will recall some principles of calculating probabilities. It is expected that they discuss various approaches and explain to each other certain misconceptions during the discussion. Students might work first on the smaller case. Once they successfully calculate the probability for five papers, they will continue to generalize. Once the formula $$p(n) = 1 - \frac{365!}{(365-n)!\,365^n}$$ is obtained, there might be different strategies for the estimation of the number n such that $p(n) > 0.5$, but it is expected that students will use technology either to calculate the number directly or to graph the function.
Formulation (didactical/ adidactical) 20 minutes	The teacher invites the students to present their findings (students are given 10 minutes for this part). The order of groups is such that the first groups to report are those that had a very concrete approach, without coming to the general function. Next, come the groups that have found the functions, with or without mistakes. Finally, groups that have concluded that $n = 23$ is the sought answer explain their reasoning.	Students write their findings on posters or prepare short PowerPoint presentations with their work and conclusions. Groups (or representatives) present to the whole class. Other students listen and compare to their own work. Students discuss, ask questions, and make sense of the arguments that all the groups presented.
Validation (didactical/ adidactical) 5 minutes	The discussion is opened after the presentations. The teacher directs the students' attention to errors in calculations and asks the students to validate and comment on the arguments of other groups. The teacher may fill the gaps in the reasoning of some groups, trying to engage the students as much as possible.	Students listen to those presenting, ask for elaboration, and comment on or discuss the suggestions on the board. It is expected that some time is dedicated to the discussion about the graph and the method of obtaining the answer $n = 23$.
Institutionalisation (didactical) 5 minutes	The teacher sums up the results and explains why they are so surprising.	Students write down the conclusions they have not reached during their work in the group.

Possible ways for students to realize target knowledge

Before coming to the main ideas, students may consider other approaches and auxiliary results, some of which will be distracting. For example, some students may engage in counting the number of pairs in a group of five:



For the calculation of the probability of having a pair of the same birthdays in a group of five people (event A), the students need two main ideas:

- The idea of calculating the probability of the *complement*, i.e., to notice that it is easier to calculate the probability that all the birthdays are different (event A^c)
- The idea of the product rule and/or the definition of the probability

The calculation then immediately follows:

$$P(A) = 1 - P(A^c) = 1 - \frac{365 \cdot 364 \cdot 363 \cdot 362 \cdot 361}{365^5} = 1 - 0{,}97286 = 0{,}02714$$
$$\Rightarrow P(A) = 2{,}71\ \%$$

Next, the students generalize to the arbitrary size of the group:

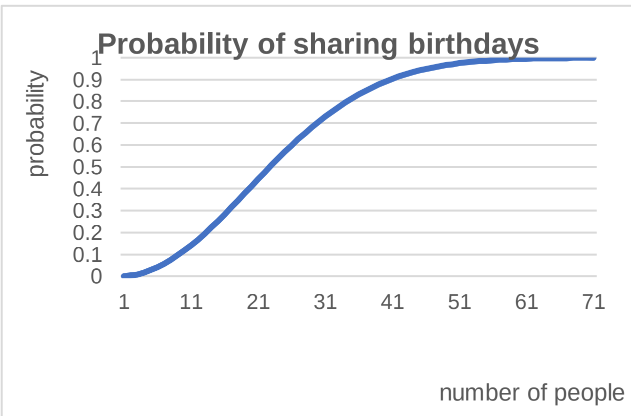
$$P(A) = 1 - P(A^c) = 1 - \frac{365 \cdot 364 \cdot \ldots \cdot (365 - n + 1)}{365^n} = 1 - \frac{365!}{(365-n)!\,365^n}.$$

For $n = 23$, we have the following calculation:

$$P(A) = 1 - \frac{365 \cdot 364 \cdot \cdots \cdot 343}{365^{23}} = 1 - 0{,}4927 = 0{,}5073 = 50{,}73\ \%$$

One should note that the first form of the function is more suitable for using the calculator. The students will encounter problems if they try to calculate 365!.

The graph of the function can be obtained using whatever tools students are used to. It can be estimated from the graph that the answer is 23, while it is also interesting to discuss the shape of the graph and its asymptotic behaviour as n increases.



NOTE: See the project's webpage for the following editable materials in Word.

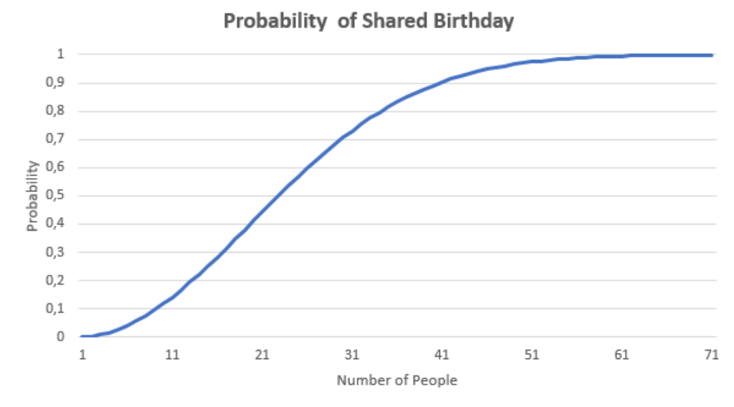
HANDOUT

TASK 1

- Write your birthday and the birthdays of five people you know but are not in class now (day and month, not year) on the prepared papers. Fold the papers and put them in the prepared bag at the teacher. Randomly select 5 papers from the bag. Check if the date is repeated on any of the tickets. Write down the finding.
- Did the result surprise you? Why?
- How many people (papers with birthdays) do you need to have, to have at least two people with the same birthday? Why do you think so?
- How many people do you need to have at least 50 % chance that two people have the same birthday? For each member of the group, write down his/her prediction.
- What is the average in the group? Why did you choose this number?
- Randomly select 23 papers from the bag. Check if any date is repeated on any of the written papers. Write down the finding. Were you surprised by the result? Why?

TASK 2

- How would you calculate the probability that two people in a group of five were born on the same date? The assumptions we will use to simplify the calculation are:
    - no leap years (we exclude 29th February),
    - assume other 365 days are equally likely,
    - assume independence of births.
- What is the probability that in a group of 23 people two people have a birthday on the same day?
- What do you think about the calculated probability? Did the number surprise you? Why?
- See the diagram below. Would you be able to write down any interesting findings?



Teachers' Inquiry in
Mathematics Education

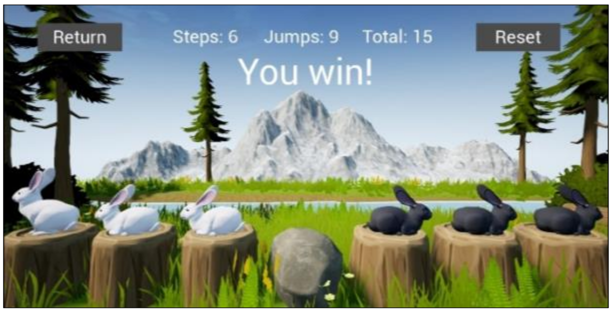
Game of rabbits

Using algebraic expressions to generalize counting number patterns

Team: Gimnazija Franca Miklošiča Ljutomer, Slovenia

Target knowledge	Target knowledge may vary depending on the students and the program of the high school. We list different possibilities under *Broader goals*.
Broader goals	Using algebraic expressions to state/generalize number patterns. Algebraic manipulations based on the distributive law. Counting and calculating sums. Problem-solving skills: Detecting patterns in a process. Exploring small/concrete cases and hypothesizing using inductive reasoning. Argumentation of own strategy in words or formulas.
Prerequisite mathematical knowledge	The scenario allows various levels of achievement, so it may be suitable to encourage students with no prerequisite knowledge to explore and hypothesize, while it may also serve to motivate more experienced students to practice their problem-solving and argumentation skills.
Grade	Age 16 (2nd grade in Slovenia)
Time	60 minutes
Required material	Game applet: Android app on Google play: https://tinyurl.com/y7b3mz7a iPhone app on App Store: https://tinyurl.com/2ztxa3c8 Black and white tokens.

Problem:
In the game of rabbits, some black and some white rabbits are standing in a line. All black rabbits stand on the left side of the line and all white rabbits stand on the right. The goal of the game is to move all the black rabbits to the right and all the white rabbits to the left of the line. Black rabbits can only move to the right and white rabbits can only move to the left. There is exactly one blank space to which one of the rabbits can move. In each move, one rabbit can make a *step* to an adjacent blank space or *jump* over exactly one other rabbit.



Play the game as many times as you like to answer these questions:
Is it always possible to achieve the goal of the game? If yes, what is the (minimal) number of moves (steps and jumps) needed in total?

Phase	Teacher's actions, including instructions	Students' actions and reactions
Devolution (didactical) 5 minutes	The teacher divides the students into groups, provides the students with the link for the game, and explains the problem. Based on the chosen target knowledge, the teacher directs the students to focus on a particular objective (e.g., counting or argumentation) and to write down their ideas.	Students download the app and play the game to understand the rules. Students ask questions to clarify their objectives.
Action (adidactical) 20 minutes	The teacher observes the students' work without interference. The teacher may recall the overall objective of the lesson if (s)he has the impression that the students play the game without the intention to observe patterns.	Students play the game until they get a better understanding of the rules and possible positions. By repeating the game, students encounter different patterns and notice which moves lead to the desired goal. Once they discover a way to achieve the goal with a fixed number of rabbits, students start to explore with other numbers. They organize the variations in a table to generalize or start thinking in more general ways to provide argumentation for their hypotheses.
Formulation (didactical/ adidactical) 20 minutes	The teacher invites the students to present their findings, hypotheses, and argumentation. Presentations may be written on posters or digitally. The teacher organizes the presentations of the groups in the order from the most concrete to the more general reasoning.	The groups work on their presentations. Some groups will write explanations in words, but it is expected that most groups use tables and formulas. Students are expected to come up with various approaches and perhaps even with different answers if they make small errors.
Validation (didactical/ adidactical) 10 minutes	The teacher leads the discussion in which students compare their results. The teacher encourages students to ask questions to better understand the arguments of the other groups.	The groups present and compare their answers. Students notice that their answers are similar or the same, but that they have reached them in different ways. Students ask questions to make connections between different approaches.
Institutionalisation (didactical) 5 minutes	Based on the students' approaches, the teacher completes their arguments to show a general answer. The teacher emphasizes the problem-solving strategies that have been used and observed.	Students ask additional questions and note the ideas of other groups. They make the connection between their work and general strategies for reasoning in mathematics.

Possible ways for students to realize target knowledge

The mobile application provides an opportunity to study the posed problem and its generalization to the case in which the number of black and white rabbits is not the same. As already the posed problem is challenging enough, we describe the details in this case. In this case, there is only one strategy and hence the minimal number of steps is the only one possible.

- *Considering small cases and generalizing based on inductive reasoning:*

  Note that the app provides the number of jumps, the number of steps, and the total number of moves. This supports the students in hypothesizing about the number of moves without knowing that there is only one strategy to exchange the positions of the rabbits.

  The students can write the numbers in a table and generalize based on the format in which they consider the given numbers. For example, based on the total number of moves, students may consider the following table:

Number of rabbits	Total number of moves
1	3 = 1·3
2	8 = 2·4
3	15=3·5
…	…
n	$n(n+2)$

  Similarly, students may consider separately the number of jumps, which is n^2, and the number of steps, which is $2n$. Then the total number of moves is

$$n^2 + 2n = n(n+2).$$

  The connection between the two ways of expressing the answer is given by the distributive law.

- *Strategy:*

  Students may write down an explicit set of moves that leads to the goal. Denote the blank space by _, the black rabbits by x and the white rabbits by o. For $n = 3$, we have the following list of positions:

xxx_ooo xxxo_oo xx_oxoo x_xoxoo xox_xoo **xoxox_o**	xoxoxo_ xoxo_ox xo_oxox _oxoxox o_xoxox oox_xox	ooxox_x ooxo_xx oo_oxxx ooo_xxx

  There are 16 positions, hence there are 15 moves. From the examples, one can conclude the following about the strategy:

  - The game cannot continue towards the goal if we have the following configurations: "_xxo" or "xoo_" So, we must avoid moving two rabbits of the same colour using steps.
  - It does not matter if we start with a step of a white or a black rabbit, the game is symmetric with respect to the colour of the rabbits.
  - We must reach the position in which the rabbits of different colours are alternating (xoxox_o) and there is a symmetry between the moves before that position and after its mirror position (o_xoxox), between those positions there are exactly 2 steps and n jumps.

- In the above example, we have alternating moves: white 1 step, black 1 jump and 1 step, white 2 jumps and 1 step, black 3 jumps, white 1 step and 2 jumps, black 1 step and 1 jump, white 1 step.

From the last observation, we may hypothesize that in general we will have $2n$ steps and the number of jumps will be

$$1 + 2 + \cdots + (n-1) + n + (n-1) + \cdots + 2 + 1 = n^2.$$

This rule can be deduced using the formula for the first n positive integers, by the method of mathematical induction, or graphically.

The described strategy is unique (up to the symmetry of black and white rabbits), but this is not obvious and could be proven by considering all possible cases of moves in each situation. We continue with a more elegant argument.

- *The direct argument for the general number of moves:*

We can count the total number of moves independently of knowing the strategy. For all the rabbits to switch places, we need that each black rabbit jumps over each white rabbit or vice-versa. Hence, there are n^2 jumps.

To find the number of steps, we consider *the total distance that all the rabbits need to pass*. This total distance is the same regardless of the order of moves. If we compare the initial and final position, each of the $2n$ rabbits should move exactly $n+1$ places from its initial to the final position. So, the total distance all the rabbits pass is $2n(n+1)$. In each jump, a rabbit passes a distance of *2 places*, while in a step it passes a distance of *1 place*. If we denote the number of jumps by J and the number of steps by S, we have a small system of equations:

$$2J + S = 2n(n+1), \quad J = n^2 \quad \Rightarrow \quad J + S = 2n(n+1) - n^2 = 2n^2 + 2n - n^2 = n^2 + 2n.$$

The case with an uneven number of black and white rabbits has two variations. In the first (original) case, there is only one blank space, while in the second case the places on which the rabbits can stand are symmetrical with respect to the middle position and there are more blank spaces behind the smaller number of rabbits. In this case, the ending position has to be exactly symmetrical to the starting position with respect to the empty space in the middle.

- *Non-symmetrical case – only one blank space.*

If the number of black rabbits is n and the number of white rabbits is m, the answer is $nm + n + m$. One can hypothesize this from the following table of "steps+jumps=total moves":

	m=1	*m*=2	*m*=3	*m*=4
n=1	2+1=3	3+2=5	4+3=7	5+4=9
n=2	3+2=5	4+4=8	5+6=11	6+8=14
n=3	4+3=7	5+6=11	6+9=15	7+12=19
n=4	5+4=9	6+8=14	7+12=19	8+16=24

The general answer can be justified as in the case $n = m$. We have $J = nm$ jumps, the total distance that the rabbits must pass is $2J + S = n(m+1) + m(n+1)$, hence the total number of moves is

$$J + S = n(m+1) + m(n+1) - nm = nm + n + m.$$

For $n = 3$ and $m = 5$, we have the following list of 23 moves:

xxx_ooooo	xo_oxoxoo	ooxo_oxox
xxxo_oooo	_oxoxoxoo	oo_oxoxox
xx_oxoooo	o_xoxoxoo	ooo_xoxox
x_xoxoooo	oox_xoxoo	oooox_xox
xox_xoooo	ooxox_xoo	ooooxox_x
xoxox_ooo	**ooxoxox_o**	ooooxo_xx
xoxoxo_oo	ooxoxoxo_	oooo_oxxx
xoxo_oxoo	ooxoxo_ox	ooooo_xxx

The strategy can be divided into three parts. In the first part, we consider only the first n white rabbits and we follow the strategy from the case $m = n$ all the way until the alternating position of n black and n white rabbits followed by the $m - n$ white rabbits, xoxox_ooo.

In the second part, we keep the black and white rabbits alternating, but gradually the $m - n$ white rabbits from the right side get involved in the alternation and $m - n$ white rabbits from the alternation pass to the left side. This part ends with the position ooxoxox_o, in which the $m - n$ white rabbits are followed by the alternating position of n black and n white rabbits.

In the third part, the block with alternating rabbits is resolved as in the case $m = n$.

Again, from the examples, one can conclude that there should be $n + m$ steps because we alternate steps between one step and a sequence of jumps. The number of jumps, assuming $n \leq m$, is

$$1 + 2 + \cdots + (n - 1) + \underbrace{n + n + \cdots + n}_{m-n+1 \text{ times}} + (n - 1) + \cdots + 2 + 1 = nm.$$

- *Symmetrical case – more blank spaces behind the smaller number of rabbits*

This is the most complicated case because there are more ways the rabbits could move and the final position can be obtained in various numbers of moves. To find a strategy with the *minimal* number of moves, one must use the moves in which rabbits jump over rabbits of the same colour as much as possible.

The optimal strategy can again be described in three parts, and the *block* consisting of n black and n white alternating rabbits is central to our description. In the first part, we come to the alternating position in the middle. In the second part, we keep the black rabbits and one white rabbit fixed, while the $m - n$ rabbits from the right side pass to the left side. In the third part, we again unravel the alternating position as in the case of $n = m$.

Here is an example with $n = 3$ and $m = 5$, where we have 25 moves:

__xxx_ooooo	_oxoxox_o_o	ooxo_oxox__
__xxxo_oooo	o_xoxox_o_o	oo_oxoxox__
__xx_oxoooo	oox_xox_o_o	ooo_xoxox__
__x_xoxoooo	ooxox_x_o_o	oooox_xox__
__xox_xoooo	ooxox_x_oo_	ooooxox_x__
__xoxox_ooo	ooxox_xoo__	ooooxo_xx__
_ox_xox_ooo	**ooxoxox_o__**	oooo_oxxx__
_oxox_x_ooo	ooxoxoxo___	ooooo_xxx__
_oxox_xoo_o	ooxoxo_ox__	

To count the number of steps in the second part we need to describe the actual sequence of steps and jumps. One way is to consider the moves required to move one white rabbit from the right of the alternating block to the left.

In each such sequence of moves, a new rabbit will jump to the position left of the leftmost black rabbit. To make this possible we need to make space by moving the white rabbits left of that position. After that, we perform $n+1$ jumps of white rabbits over the fixed rabbits. Finally, we need to move the white rabbits on the right side to the position immediately right of the right end of the block so that they can start jumping over the fixed rabbits.

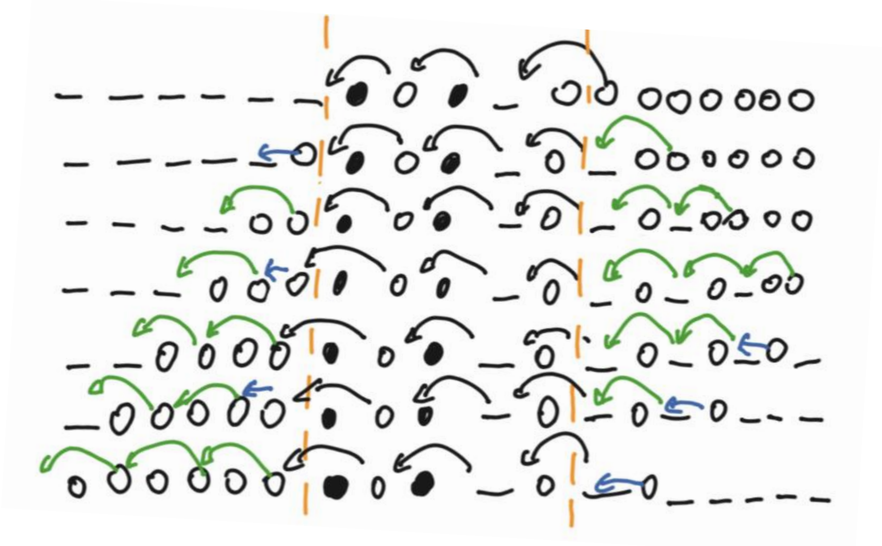


Figure: Illustration of the second part of the strategy in case $n = 2$ and $m = 9$.

When there are many white rabbits, we notice that the number of moves needed to come to the left of the block is:

- 1 step for the first rabbit
- 1 jump for the second rabbit
- 1 jump and 1 step for the third rabbit
- 2 jumps for the fourth rabbit
- 2 jumps and 1 step for the fifth rabbit
- 3 jumps for the sixth rabbit
- etc.

So, the number of moves to come to the left of the block is

$$1 + 1 + 2 + 2 + 3 + \cdots$$

and there should be exactly $m - n$ numbers in the sum. We see that the result depends on the parity of the number $m - n$.

We can also describe the number of moves to come to the right of the block:

- 1 jump for the first rabbit
- 2 jumps for the second rabbit
- 3 jumps for the third rabbit
- ... and this goes on, taking every other rabbit, until the last rabbit for which we have $\frac{m-n}{2}$ or $\frac{m-n-1}{2}$ jumps, depending on the parity of $m - n$. After that, we deal with the remaining rabbits, for which we have:
- 1 step and $\frac{m-n}{2} - 1$ or $\frac{m-n-1}{2} - 1$ jumps
- ...
- 1 step and 2 jumps
- 1 step and 1 jump
- 1 step

So, the number of moves to come to the right of the block is

$$1 + 2 + 3 + \cdots + 3 + 2 + 1$$

and there should be $m - n$ numbers in the sum.

The total number of moves needed to move $m - n$ white rabbits from the right side of the block to the left side of the block is

$$(1 + 1 + 2 + 2 + 3 + \cdots) + (m - n) \cdot (n + 1) + (1 + 2 + 3 + \cdots + 3 + 2 + 1).$$

This is the number of moves in the second part and the closed formula for this expression is

$$\frac{m^2 - n^2 - 1}{2} + m - n \quad \text{if } m - n \text{ is odd,}$$

or

$$\frac{m^2 - n^2}{2} + m - n \quad \text{if } m - n \text{ is even.}$$

We know from the case $n = m$ that the total number of moves in the first and third parts of the strategy is $n^2 + 2n$, so for the total number of moves we get

$$J + S = \frac{m^2 + n^2 - 1}{2} + m + n \quad \text{if } m - n \text{ is odd,}$$

or

$$J + S = \frac{m^2 + n^2}{2} + m + n \quad \text{if } m - n \text{ is even.}$$

We can come to this number also without knowing the general strategy, but in this case, we can have many jumps and it is easier to understand the minimal number of steps that are needed. The crucial understanding comes from the fact that the black rabbits can move only in the $2n + 1$ places in the middle. This part is the part in which we have positioned the alternating block.

Only white rabbits can occupy positions to the left and to the right of that block. When a rabbit jumps it moves exactly 2 places, so if we think of all the places as odd or even, we notice that a rabbit on an odd position cannot reach an even position without making a step, and vice-versa. Each white rabbit must come to the place immediately right of the block before jumping over the block and it comes to the place immediately left of the block.

So, about half of the $m - n$ rabbits need to move by one step on the right side and about half of the $m - n$ rabbits need to move by one step on the left side. Of course, the number of steps needs to be an integer, so if $m - n$ is even we have $\frac{m-n}{2} + \frac{m-n}{2} = m - n$ steps, while if $m - n$ is odd we have $\frac{m-n-1}{2} + \frac{m-n-1}{2} = m - n - 1$ steps.

In addition, we have shown that there are $2n$ steps in the case $n = m$, so the total number of steps is $S = m + n$ if $m - n$ is even and $S = m + n - 1$ if $m - n$ is odd.

The total distance that the rabbits need to pass from the initial to the final position is

$$2J + S = m(m + 1) + n(n + 1).$$

From this, we can calculate the number of jumps as

$$J = \frac{(2J + S) - S}{2}.$$

Once we do the calculation, we get the above result $J + S$ of the total number of moves.