Teachers' Inquiry in
Mathematics Education

TIMEless
Practice Reports

A compilation of reports about
Lesson Studies implemented
by the teams of teachers

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TIMEless Practice Reports

A compilation of reports about Lesson Studies implemented by the teams of teachers

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http://time-project.eu

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Teachers' Inquiry in Mathematics Education

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Introduction

The main aim of project TIME was to promote Lesson Study as a way of teachers' professional development and to support teachers in designing innovative inquiry-based mathematics lessons. As is mentioned in the document *TIMEless: A short introduction to Lesson Study – TIMEless ideas for professional development* (Bašić, 2020):

> *There is a striking similarity between Lesson Study as an activity for teachers, and the experience aimed at students in inquiry-based education: namely, the principle that you learn from studying a problem through experimenting with hypothetical solutions. For teachers, problems are related to students' learning (with specific and more general goals), and you keep finetuning your experiment until you are ready to share your findings with others (p.12)*

This process of inquiry is not limited to the context of Lesson Study. Every process of design should be a process of inquiry in which one visits many different stages

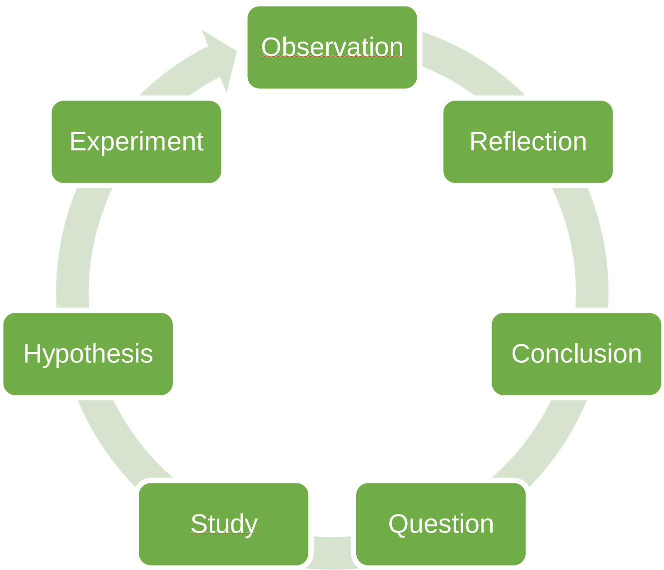

Figure 1. The inquiry cycle

In Lesson Study, a team of teachers jointly investigates a didactical issue by experimenting with variations of implementing a certain a mathematical topic in the classroom. The investigation starts with the study and planning, then follow the design and implementation, and at the end, the team reflects on their observations and writes a report to share their findings with the wider educational community.

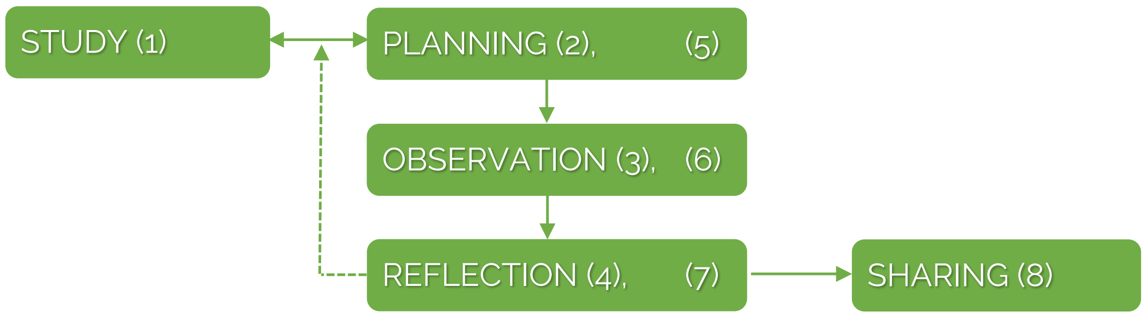

Figure 1. Phases of Lesson Study, taken from (Bašić, 2020), an adaptation of a diagram by Stigler and Hiebert (1999)

A report has a simple form, but carries the most important information that enables other teachers to learn from the described study and to use the design of the lesson as a source of inspiration. Reports in general provide earlier experience and motivation for the theme, a clear statement of the goals pursued in the Study or the experimental hypotheses made by the team, a problem given to the students, observations made by the team members, conclusions from the reflection session, future perspectives and references. In the project TIME it was agreed that a report consists of obvious four sections:

Identifying the problem and learning goals

Planning and creating the lesson plan

Observation of the Study Lesson

Reflection and concluding remarks

In this booklet, we present 19 reports based on the Lesson Studies implemented by the nine teams of teachers involved in the project TIME. These reports have been translated into the languages of the teams and published in national educational journals. For the majority of the reports, there is also a full scenario that other teachers can follow to implement the lesson themselves. Scenarios are published on the project web page.

We have ordered the reports according to countries and teams. In the Danish *Midtsjællands Gymnasium,* there were two teams, one from Haslev and one from Ringsted. The team from Haslev started with a very elementary fact about the multiplication of integers that students learn by heart, but have difficulties explaining. We chose this report as the opening of this booklet as it is the least demanding topic, but exhibits the main idea that all the teams try to achieve – to discuss mathematics in depth and make it relevant to the students. The second Study of this team is almost on the other side of the spectrum of secondary mathematics education, as it motivates students to try to discover the concept of the derivative in the appealing context of a rocket launch.

The team from Ringsted presents three Lesson Studies, all focusing on the modelling and application of mathematics. In the first, the focus is on various ways of making predictions and using linear regression. The second enables students to be creative while designing a ski slope with piecewise functions. In an almost effortless way, it builds students' intuition about more advanced concepts like the continuity of a function. The third puts the focus on the students' difficulty in producing an algebraic expression to be used in optimization. Each of the three asks the students to use functions in a different way, while the third one also served for the production of a video introducing Lesson Study to newcomers.

Next, we have the Lesson Studies of four teams from Croatia. There were two teams from *XV. gymnasium* and two teams in *V. gymnasium*. One team has produced three intertwined Studies with the same theme – the square root. It shows a long and very interesting journey in which the various aspects of the square root are discussed. First, the square root of a number and the convention that it is never negative, next the square root function as an inverse of the quadratic function, and finally, an application showing that some situations in life are modelled by the square root function. Students are engaged by very diverse didactic materials such as a new colourful board game and a video. The team has also produced a video describing their experience during all three Studies.

The second team from Croatia connected their two scenarios by a way of work. The two Studies are based on a very dynamic group work that encourages students to listen to each other and use other people's work to come to a better understanding of mathematics and how it is used in science. Motivated by the pandemic and the earthquake in Zagreb, the first Study dealt with the importance of the logarithmic scale. The second Study puts the students in a completely new and creative role of building concrete physical models of geometrical concepts and based on them improving their mathematical language.

The next team focused on the important topic of argumentation and making sense of mathematics. In the presented Study the students were given geometric pictures which describe a rule or a theorem known to students from algebra. In this way, the students connected to areas of mathematics, discovered patterns, and gained confidence in their own use of mathematical arguments.

The fourth team has tackled two aspects that they have found very challenging for their students. One Study can be very closely connected with a few Studies mentioned so far. It deals with mathematical reasoning and the way the students learn how to prove in mathematics. The topic is the principle of mathematical induction and the emphasis of the Lesson Study is on preventing typical students' errors. It also brings focus to the ability to read and use the textbook, a skill we somehow unjustifiably neglect. For the second Study, the team has invented a new didactic tool – the radian-meter. Its role is to support students in understanding the definition of trigonometric functions and its relation to geometry.

Very interestingly, the Dutch team from *Utrechts Stedelijk Gymnasium* has also chosen the theme of trigonometry as the centre of their Lesson Studies. They exchanged experience with the Croatian team and developed two more interesting scenarios. In the first one, the definition of the trigonometric functions is connected to their graphs by using yet another tool that the students can hold in their hand – this time, a simple CD with a dot sticker. The second connects trigonometry with argumentation and discovery of new rules. Inspired by a very well-known formula that is hard for students to prove, and the very much enjoyed technique of origami, the lesson inspires students to engage in a mathematical proof of a formula based on the folding of a paper.

In Slovenia TIME gathered teachers in two schools, *Gimnazija Franc Miklošič Ljutomer* and *Gimnazija Jesenice*. Both teams present two Lesson Studies. In Ljutomer, the first Study has created a simple yet very rich story of a student visiting his grandma and looking for a jam. Unfortunately, there is no jam and the granny blames the frost. The students need to search for data to explore this relatable situation. On their way, they will encounter the need to study many basic properties of functions and in this way realize the hidden goal of the teachers. In the second Study, we once more see the connection between geometry and algebra, but this time with the goal to calculate well-known sums based on another familiar context.

In Jesenice, we also have two contexts that are very close to the students. The roof of a warehouse is yet another source of inspiration to engage in the process of modelling and studying spatial geometry, leaving a lot of space for the students to be creative and choose the functions they will use to model the roof. The final Study is one that carries a wow effect. It is based on a famous probability paradox, which captures the students' attention from the beginning and makes them remember how lightly they have grasped the difficult or deep mathematical ideas such as considering the complement, understanding the asymptotic behaviour of the exponential growth, and exploring the graph of a new function using technology.

Minus and minus gives plus

Making sense of a principle learnt by heart

Flóvin Tór Nygaard Næs

Midtsjællands Gymnasium, Haslev, Denmark

Defining the problem and target goals

When students start high school, they arrive with a lot of knowledge from their time in primary school. Some of the mathematical knowledge that students have obtained needs to be adjusted and made more precise. One example of this is the phrase: "minus and minus gives plus" that students often refer to when they are doing mathematical work involving negative numbers.

Despite knowing this rule, or maybe indeed because of the unprecise formulation of the rule itself, we as teachers often experience students stumbling over calculations with negative numbers. This was our motivation for planning a lesson for first-class students of Midtsjællands Gymnasium, where the focus was on the students' own interpretation of the title phrase.

Planning and creating a lesson

While planning the lesson we did not take a good practical example that could force the students to speak about and explain why the product of two negative numbers must be positive. Instead, we chose to challenge the students to come up with an explanation themselves for the rule "minus and minus gives plus" and to give an argument for why the rule works.

Observation

At the start of the lesson, the students discussed the phrase in small groups. One of the groups reformulated the rule as follows: "*When you multiply and divide two negative numbers, then it automatically becomes positive*".

Next, the groups were encouraged to formulate an argument for the rule that $a \cdot b > 0$ if $a, b < 0$. As expected, the students found this challenging and their reactions varied. Some students kept on working, while others became more passive. It was quite obvious that the students were not used to work with mathematics in this manner.

$$^{-}3 + ^{-}2 \blacktriangleright ^{-}5$$
$$^{-}3 \cdot ^{-}2 \blacktriangleright 6$$
$$^{-}2 - 3 \blacktriangleright ^{-}5$$
$$\frac{^{-}3}{^{-}1} \blacktriangleright 3$$

Figure 1. A calculation done by CAS.

The groups came up with a lot of different mathematical explanations. Many groups used their CAS tool to perform calculations such as the one presented in Figure 1. Others tried to utilize the CAS tool to give a graphical explanation, but they found it difficult to give a good explanation.

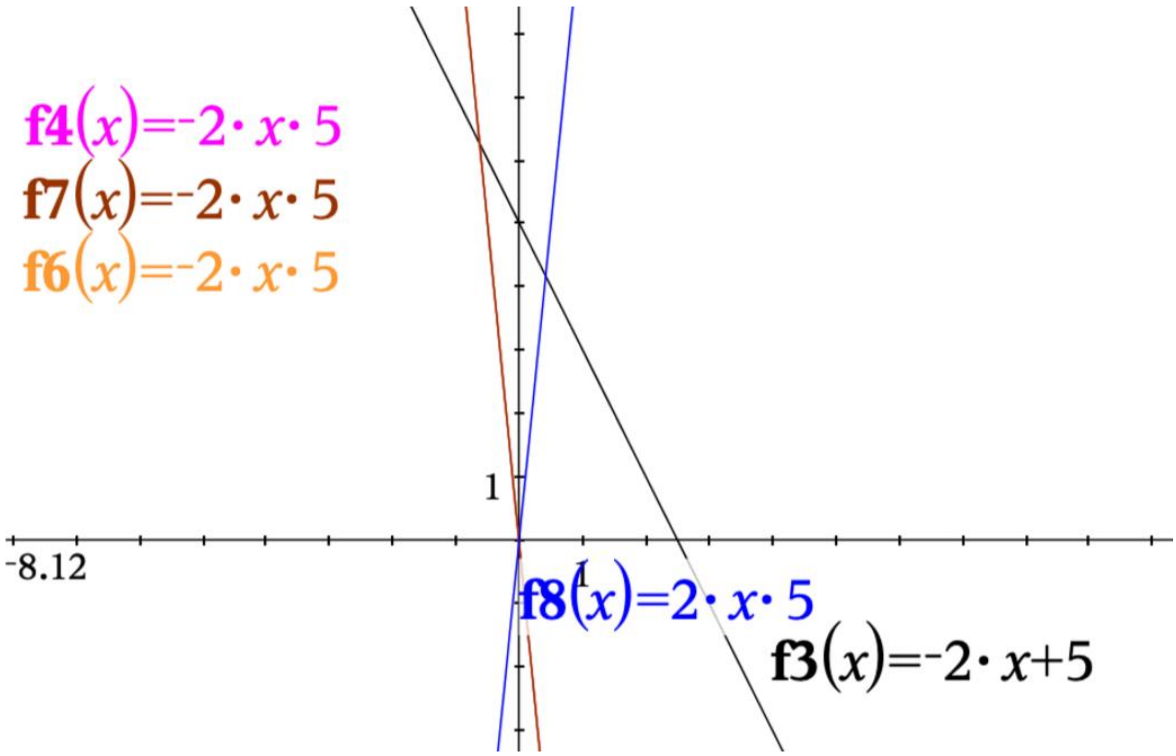

Figure 2. Graphical representations of multiplying with negative numbers.

Many students preferred to use oral arguments or examples such as "*To not not rest is the same as to rest*". When the students were asked what these examples had to do with the multiplication of negative numbers, they typically replied "We have learned this in primary school" and followed it up by listing some rules they remembered.

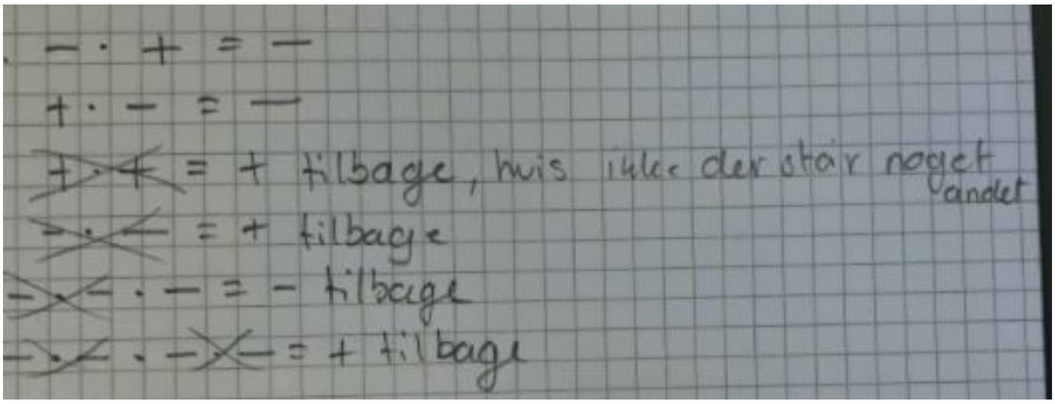

Figure 3. Rules students remember from primary school.

During the lesson, it became obvious that the students had no inner need for a further rational argument why the multiplication of two negative numbers is positive. They knew this beforehand! It was enough justification for most of them that examples from their calculator verified the rule while others interestingly justified the rule by referring to their teacher in primary school as a source of truth. As time went by and the students realized that in high school we are looking for other types of explanations and arguments, they became more and more frustrated. Some of the students resigned and now demanded an explanation from another authority in form of their new high school teacher. Or maybe they were curious to learn more about mathematics and new methods previously unknown to them.

At the end of the lesson, the teacher gave two different kinds of explanations on the blackboard, one of which was a graphical explanation combining some of the ideas students had shown earlier in the lesson. Looking at the expression $y = -2x$, a table was lined up and for some positive values of x the corresponding values of y were calculated. For the students, it was not surprising to see that the corresponding points in the fourth quadrant were laying on a straight line. Intuitively the students found that the line must continue in the second quadrant, with the points consisting of negative x-coordinates and positive y-coordinates.

As a supplement to this rather graphical justification of the rule, the teacher also gave an arithmetic example using the definition of a negative number and the distributive law.

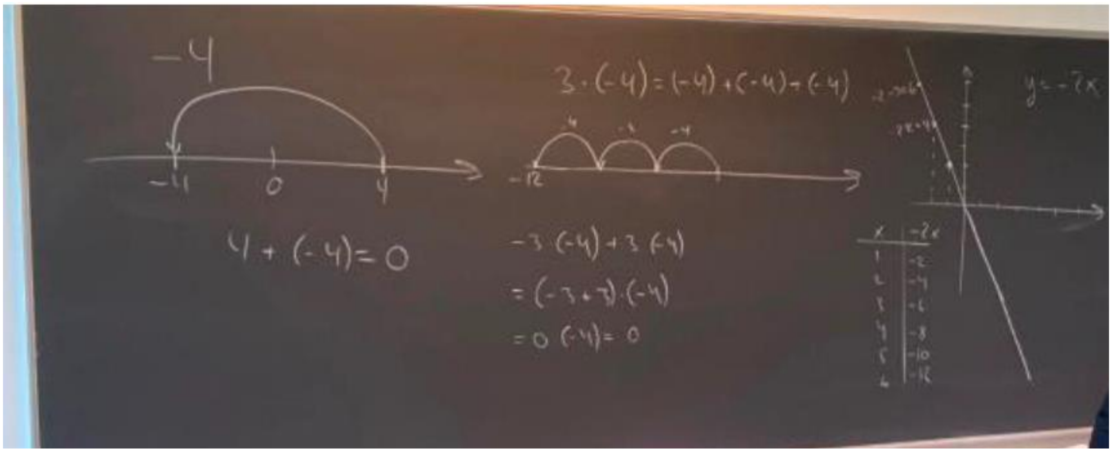

Figure 4. Blackboard explanation of the teacher.

Reflection

After the lesson, we concluded that only a few of the students had been able to understand the second argument and many students probably left the lesson with a lot of unanswered questions. It would have been very interesting to get one of the students to repeat the arguments and formulate it on their own, but time was unfortunately up.

Finally, it should be mentioned that this lesson was held in two different classes. For the second run of the lesson, we decided to start and end the lesson with three small exercises of the type $(-2) \cdot 4$, $6 \cdot (-3)$ and $(-4) \cdot (-3)$. The purpose was to get to the point more quickly and to focus more on the justification of the rule. As it turned out, the multiplication of two negative numbers did not seem to give the students big problems, at least not when the exercises were presented this way. So, in the second lesson we won some time, but also lost some interesting discussions regarding which operations are actually referred to in the phrase: minus and minus gives plus".

Rocket launch

Modelling speed as a motivation for derivatives

Flóvin Tór Nygaard Næs

Midtsjællands Gymnasium, Haslev, Denmark

Identifying the problem and target knowledge

One of the subjects that students find notoriously difficult in the second year of high school in Denmark is differential calculus. Hence, the focus of this article is on a lesson that is placed before the students really learn about the differential quotient and other related concepts. One of the main goals of the lesson is to motivate the students before the going gets hard. For the teacher, the hope is that it will be possible to link the experience from this lesson with the mathematical notions at a later stage.

Planning and creating the lesson plan

The students were given a video sequence of a Space launch and were asked to calculate the average speed of the rocket in the first minute after the launch. In addition, they were asked to estimate the speed of the rocket after precisely one minute. Mathematically the goal of the lesson was that the students should be able to distinguish between average and instant speed.

Figure 1. A screenshot from the video showing a rocket launch

Observation of the Study Lesson

Our original idea was only to pose the second question, but we also wanted to make students think about the average speed and the difference. We witnessed, while the students were calculating the average speed during the first minute, that they had a lot of valuable mathematical discussions in the groups. For example, they discussed about what data should be used and which units were relevant.

It was interesting that many students were thinking more like in a physics lesson and not using mathematical formulas. Here is one example of the students' reasoning:

- *After 60 seconds the rocket flew 8,7 km, which is 8700 m*
- *We have divided 8700 m with 60, which is how long the rocket was flying expressed in seconds*
- *8700 / 60 = 145, so the average speed is* ***145 m/s***

Although there is not much wrong here, it was surprising for us that the students did not use the formula for the slope $a = \frac{y_2 - y_1}{x_2 - x_1}$ and they did not see the connection between this formula and the average speed. Probably this was one of the reasons for the difficulties they had in estimating the instant speed after one minute.

For the second run of this lesson, we decided to recall the formula with the students beforehand. While this recollection generally made the students more capable of approaching the more difficult second question, this interestingly also led some students to more confusion and misunderstandings about the easier first question.

For example, one group of students mistakenly calculated the average speed between the 30th and the 60th second as the average speed during the whole first minute.

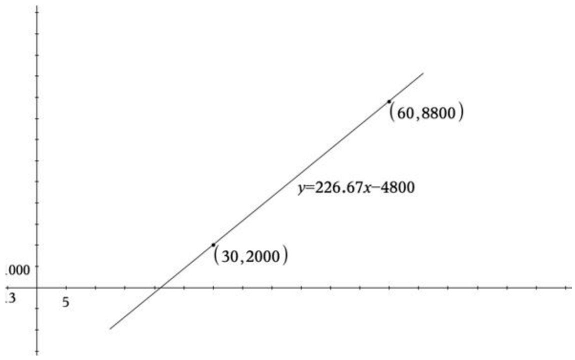

Figure 2. Students calculating average speed using a linear model

Another group calculated the average of six average speeds.

Efter 10 sek: $0.2 \cdot 60 \blacktriangleright 12.$

Efter 20 sek: $0.8 \cdot 60 \blacktriangleright 48.$

Efter 30 sek: $2 \cdot 60 \blacktriangleright 120$

Efter 40 sek: $3.8 \cdot 60 \blacktriangleright 228.$

Efter 50 sek: $6 \cdot 60 \blacktriangleright 360$

Efter 1 minut: $8.8 \cdot 60 \blacktriangleright 528.$

Gennemsnitlig fart: $\frac{12+48+120+228+360+528}{6} \blacktriangleright 216 \text{ km/t}$

Figure 3. Calculating the average speed over a minute as an average of six averages over 10 seconds

In general, these mistakes were "good" in the sense that in the validation phase it was obvious that we should focus on this type of misunderstandings. For the teachers, it was also our own plan to stay in this validation phase for as long as possible and hopefully get many students' arguments for and against the different approaches. This can be quite a challenging phase, but also potentially a very fruitful one. The challenge is to formulate questions instead of answers, so that the students themselves can formulate arguments that validate or reject their work.

	A sek	B meter	C	D	E	F	G
=				=QuadRe(			
1	0	0	Titel	Andengr...			
2	10	200	RegEqn	a*x^2+b...			
3	20	800	a	2.60714...			
4	30	2000	b	-8.2142...			
5	40	3900	c	-14.285...			
6	50	6200	R²	0.99946...			
7	60	8800	Resid	{14.2857...			
8							
9							
10							

Figure 2. Students' work using quadratic regression

Regarding the second question, it was interesting to watch all the different types of approaches of the students to calculate an estimated velocity of the rocket after one minute. Many groups collected data and made different kinds of regression, and tried to use the methods they have learned earlier, but they had problems utilizing this to answer the posed questions.

One group even drew a tangent line, but did not use it and had to give up on answering the second question. Again, the presentation of this kind of student work is valuable as it opens up the opportunity to discuss the interesting use of mathematics.

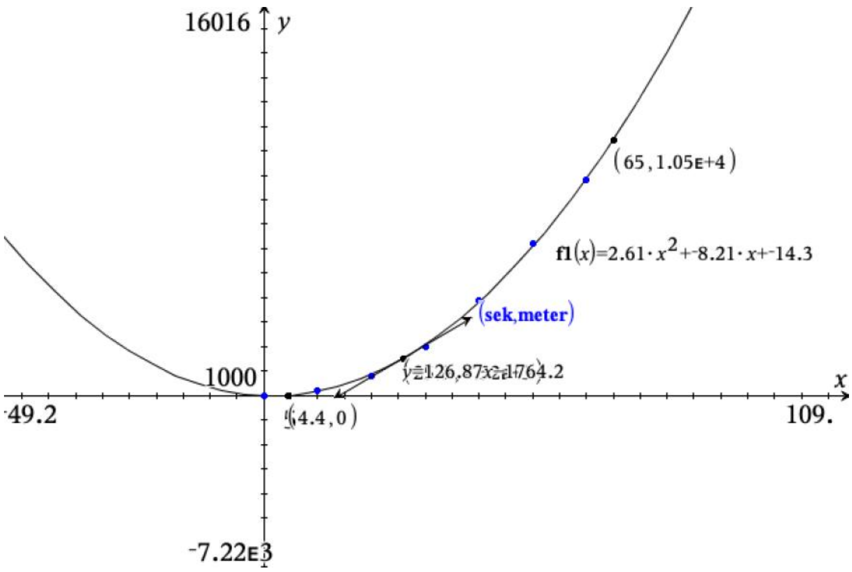

Figure 3. Graph of the function with a tangent line

0-10: 00.2 km
200/10=20 = 20 m/s

10-20: 00.8 km
600/10=60 = 60 m/s

20-30: 02.0 km
1200/10=120 = 120 m/s

30-40: 03.9 km
1900/10=190 190 m/s

40-50: 06.1 km
2200/10=220 220 m

50-60: 08.7 km
2600/10=260 260 m/s

Figure 4. Average speeds every 10 seconds

We can also mention a couple of the groups that were actually able to present an estimate of the speed of the rocket after one minute. One group calculated different average speeds in 10-second intervals. Essentially, this group's answer to the second question was the average speed in the last 10 seconds just before a whole minute had elapsed. Another group adapted a similar strategy, but instead chose to calculate the average between the average speed in two intervals, the interval from 50 to 60 seconds and the interval from 60 to 70 seconds. These inputs were interesting to discuss with the class and the obvious follow-up question is: If you had another go at the question how would you approach it to get an answer as precise as possible? In the lessons that followed we often revisited this classroom discussion when we were learning new concepts such as the secant line, the tangent line, the differential quotient and so on.

Reflection

Figure 5. Classroom during a Study Lesson at MSG

On a final note, we can mention that beforehand we, the teachers, were discussing to which degree the students would be critical of the mathematical model itself and if this could steal some of their focus. In the end, we can conclude that this was not a big issue for the students, as they do not use much time discussing the rocket's direction or the precision of the data collection based on the video. Students were engaged in the problem because of its interactive nature and actually gained a valuable piece of experience that they could connect to their future mathematics education.

How many apples are there in a pile?

Discussing averages and the optimal linear model

Jeanette Marie Axelsen

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Identifying the problem and learning goals

For the students, the notion of the residuals, the residual plots and the least squares method is only a command within a Computer Algebraic Systems (CAS) that is considered a black box, while drawing the optimal line in a data plot and understanding what is meant by 'optimal line' is often lost. So, the target knowledge for this lesson, planned and trialled in two first-year classes at MSG Ringsted, was to make the notion of the optimal line closer to students. A more fundamental goal was to make students argue about averages. Besides getting the students to reach a way to evaluate the linear model by the vertical distance between the observed data and the value in the model, the goal was to get the students to draw a well-dimensioned coordinate system and plot the points of data from a table. This is not easy for many of our students in Ringsted, so the secondary goal for the lesson was to practice this.

Figure 1. Pile of apples

Planning and creating the lesson plan

The students were presented with the following problem:

At the Roskilde Market, there is a stand from Pi Brewery, where a competition has been set up that involves guessing how many apples are there in the pile, knowing that the weight of all apples in the pile is 100 kg. The one who comes closest to the correct number is the winner of a new bike. Information about the apples is given in the following table:

Name	*Dennis*	*Mette*	*Lotte*	*Bo*	*Kaj*
Number of apples	*5*	*10*	*15*	*20*	*1*
Weight (gram)	*1000*	*1100*	*2400*	*2500*	*100*

The students are asked to answer the question without any use of CAS, but they are invited to use a calculator, as well as paper and pencil. After 10 minutes they will make a paper presentation about their estimates and reasoning for the rest of the class. After the discussion about mean averages, the students are invited to use the same data and find what would be, in their opinion, the optimal line that describes the scatter plot.

The team of teachers spent most of the time in the planning phase to test different data, so the students could have some obvious different approaches such as taking the mean average of all the apples or using one apple (note that for Kaj this is written explicitly) to get an estimate of the weight of all the apples, or to determine the slope of the straight line through two different points from the table and to put $b = 0$. The team considered using the data about apples or M&M candies bought in a supermarket, but it became clear that it would be more useful to construct the data on our own. The apples bought in the supermarket all had the same weight and the M&M's also did not have enough variation. So, the team ended up with the data shown before. Now the team had to find a rich context where it would be needed to find an estimate for the weight of an apple based on a set of data and at the same time pose an open question, so that the students were given the possibility to work inquiry-based.

Observation of the Study Lesson

During the trials of the lesson in two classes, it became clear that the style in which the lesson was tested would affect the students' approach to the problem. At first, the lesson was planned to be settled in the class 1.y two weeks after the trial in 1.x and with an intermediate period of time to adjust the lesson based on the reflection meeting after the first round of the cycle. Unfortunately, because of Covid-19 the class 1.y was sent home for a week, and the lesson was postponed for another two weeks. At this time the class had started a new

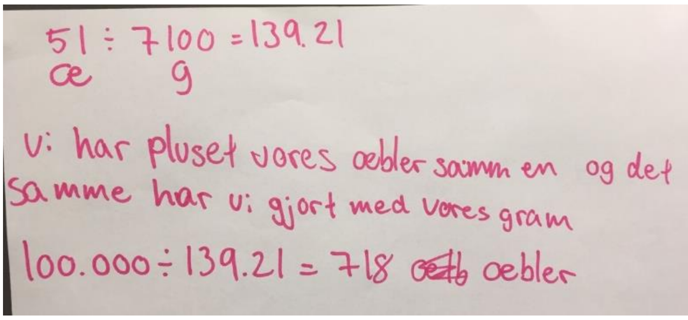

Figure 2. Translation: we have added our apples and we have done the same with our grams.

subject, descriptive statistics, and several of the approaches observed in 1.x were not even tried in 1.y. Here all the focus was on the mean average.

Another detail that had an impact happened after the first test. The team agreed that the students should have access to the grid paper from the beginning. The teacher forgot about this agreement in the trial with the class 1.y, so it could be another reason why the students did not get the idea of using regression at all. In addition, up to that moment, the students have had math lessons in which they had worked on statistics and because of Covid-19 they had to leave the classroom to get a larger one with plenty of room for all students, teachers and observers. In connection with the change of classroom, several students had asked if they should bring computers and received the answer that it would not be necessary for the coming lesson. So, because the students are used to do linear regression using CAS tools, it was not obvious to them, that they could use regression with paper and pencils.

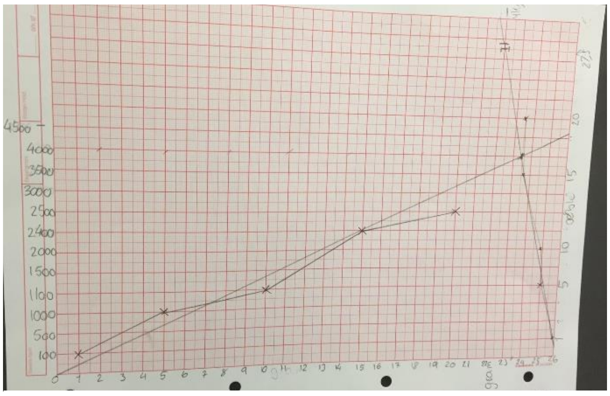

Figure 3. Example of student work showing the linear model.

In the second part, it was very ambitious to expect that the students will reach the method of least squares, especially when considering the time they have spent drawing the coordinate system and finding the line through the points. On the other hand, the group discussions among the students were interesting – for instance, they discussed how to scale the coordinate system and what to put on its axes. They have correctly reasoned most of the time, and what they have learned in this process is important. The dynamics in several groups, both in the first and the second trial, did not always result in the most precise mathematical argument, especially if it did not come from the most proficient student in the group.

In the formulation phase in the first round, the following argument was seen: "*This looks the simplest, so this must be the right answer*," or "*This looks the most advanced in a* *mathematical way, so this must be correct*." For the students, it is not about understanding what the others are presenting, but what they feel is the right answer. Perhaps it is very representative when a student says to one of his peers in the group: "*It's just math. We are not meant to understand*."

The motivation for finding out how many apples there are in the pile gives a high level of engagement among the students. One student exclaimed: "*Is there at all a correct answer?*". This is a very interesting question because the answer is that we can come up with several estimates for the weight per apple in different ways, but it will always be an estimate. The only way to get the right answer is to count the apples in the pile! So, talking about the best estimate of the number of apples in the pile in this way would be a good starting point for talking about modelling in general.

Reflection

The lesson has given an opportunity for a follow-up lesson about what it means that the sum of distances from data points to a given line should be minimized. Most people know that the shortest distance is perpendicular to the line, so why do we define the shortest distance as the vertical difference between the data point and the given line? And why do we square the residuals? Why not just take the absolute value of the distance? One student also noted that when we use regression, the constant b is non-zero. In this way, the linguistic representation of the constants is challenged by modelling. And why should not the best straight line be the one with $b = 0$?

When Team-Ringsted prepares for the next cycle, the focus will be on the importance of the context as a strong motivation to get the students to find an answer and arouse their curiosity. We would like to take a closer look at how we can get students to formulate answers to an even greater degree, so that the teachers only support the students in the inquiry process. As one of the guest observers noted, the teacher always made the resume for the students' presentations. Why not ask one of the other students to repeat what was said? This can also force or demand a culture where students need to understand the thinking behind the other students' answers or presentations, but it can also strengthen their ability to talk to each other about math using math language.

Ringsted Hill

Constructing a piecewise linear function

Jeanette Marie Axelsen

Midtsjællands Gymnasium, Ringsted, Denmark

Identifying the problem and target knowledge

In this report we present the second of the four cycles of Lesson Study in the Ringsted team of Midtsjællands Gymnasium implemented in December 2020. The main intention of the design was to create an inquiry-based mathematics situation in which the students will investigate and discover a piecewise-defined function. Part of the target knowledge is to use the mathematical notation for the functional expressions, while the broader goals are to recall graphing of the elementary functions and gain intuition about continuous and differentiable functions.

In this Lesson Study, we also wanted to investigate to what degree we would see different approaches, and to compare learning outcomes of the lesson between the first and second grade A-level students. None of the classes had previously met piecewise-defined functions, but the class 2.g had been taught all types of elementary functions as well as an introduction to differential calculus, while the class 1.g had only learned about the linear and exponential functions.

Planning and creating the lesson plan

The problem for the students is described in the following way:

> *The company "Curves for Everyone" has been commissioned to make a parking garage in the buildings at the old mill in Ringsted. Inspired by Copen Hill (figure to the right), there will be a new attraction in Ringsted, namely a ski slope. The building is 35 m high, and the hill extends 45 m from the wall of the parking garage. How do you think this Ringsted Hill should be designed?*

Figure 1. The parking garage in Ringsted and the ski slope Copen Hill.

The goal of the task is to describe a ski slope using elementary functions based on a sketch that students had drawn on graph paper. In front of them laying on the tables was a catalogue of options for their design. Likewise, it was intended, but forgotten, to provide a guide for the CAS tool explaining how to work with the sliders, which might support writing a functional expression, using regression or obtaining a mathematical description of the ski slope in some other way.

Observation of the Study Lesson

The students in 1.g, in which the first trial of the lesson was completed, designed a slope which clearly consisted of several types of functions. The discussion between the students was about the speed at the top and the middle, so that the speed could be reduced before the skier reaches the bottom. Although the sizes of the hill were discussed as unrealistic, and the local students were aware that the cemetery on the other side of the road would cause problems, the challenge of the task of creating a ski slope with different slopes along the way and without breaks and jumps was accepted.

In the first grade, students had difficulty achieving the goal of writing the functional expression. Some of this may be due to the way that the didactic environment was transferred to the students. Several of the students had read points on the graphs on their sketches, so they would have been able to get to the answer using the two-point formulas for the exponential or the linear function.

In the second trial a week later, in the second-grade class, we changed the lesson plan so that there would be a clearer line between the design part of the ski slope and the actual mathematical description. The students were presented with both a catalogue of functions and the instructions for Ti-Nspire, which were forgotten in the first trial. While the first graders would often resort to paper and pencil, it is characteristic for the second graders that they have almost forgotten paper and pencil and prefer working with Ti-Nspire. As a consequence, it turned out that the second graders had a hard time finding an appropriate expression for the ski slope, and they have chosen one function for the whole slope rather than dividing it into parts. Discussing and designing the slope with paper and pencil first would have certainly helped the students here.

The students in the groups sat around one person who had a computer, as opposed to the first-year students, where several students were working on the design at the same time. When using the technology became too difficult for the students, it became a hindrance and students' focus shifted from the task to the technical aspects, which led to a time issue. On the other hand, it was clear that the second-year students had a significantly richer academic vocabulary as they could talk about asymptotes, the continuity and smoothness of the graph, and in this way, they spent more time in their mathematical development.

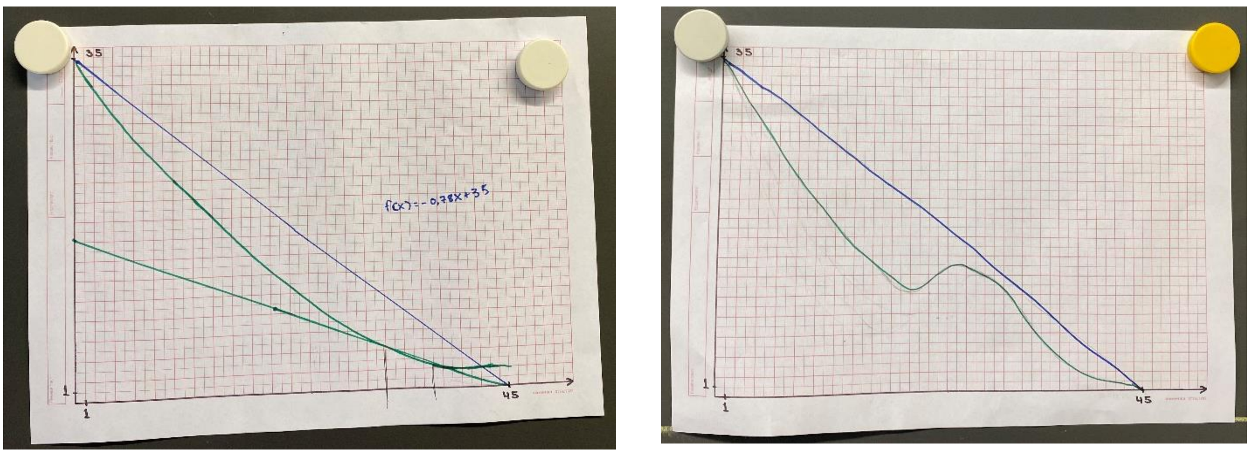

Figure 2. The design of the second year (left) and the first year (right) students. The blue curve is a draft.

Overall, it was surprising that the first graders went further towards the target knowledge of the lesson, namely an understanding that functions can be piecewise-defined and how to determine an expression of the function. In our team, we subsequently discussed that it could be an idea to try out a third round, which then could not be in an A-level class, but which matched the second graders in the way that they would have been through the elementary types of functions and calculus.

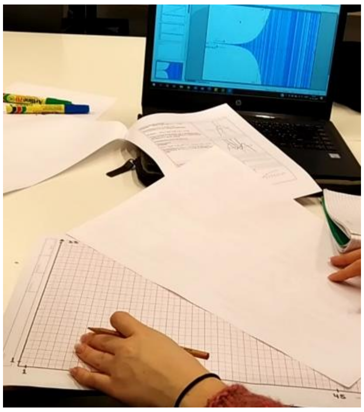

Figure 4. An example of working with sliders with the confusion of students not familiar with such experiments. Notice the blue area on the screen, which is a result of an inappropriate use of sliders.

In the second round, we also changed too many didactic variables in relation to being able to make a clearer comparison of the two lessons, such as the previously mentioned guides that were available to the second-grade students. However, some of the changes solved issues such as the need for better communication in the presentation of students' products to the whole class. In the second trial, we divided the board, so the students knew exactly where to hang their sketches. Also, we had magnets to hang their sketches, so it became much easier to take the sketches up and down again for the second round of presentations, in which the students also had to describe their ski slopes mathematically. Using two different coloured markers also made it clear to everyone what was the group's first and second draft. Ideally, the division of the board into 6 fields would also have given space to write the expressions, so it would become clear what was the connection between a sketch and an expression. From here the teacher could at the end have gathered the students' work together towards a summary of the lesson and the mathematical expression for a piecewise-defined function. Since too many parameters were changed between the two trials, it would be good to try out the lesson once more. Unfortunately, we had to abandon this idea due to the lockdown.

Reflection

For further investigation, it would be exciting to not use the help sheets for the students. The main idea was that if students are used to working with sliders, regression or other tools, then the sheets will act as an aid to their memory. However, if the techniques have not been used to a significant degree, then the students can end up with technical problems that will shift the focus from the task itself, and that would be inappropriate in relation to the purpose of the lesson.

Concerning the hypothesis of the differences between introducing the piecewise-defined functions in a first and a second year, we must conclude that we can easily introduce the subject to a first-year class with a high learning outcome. At the same time, it could also create a basis for a discussion about continuity as well as differentiability later in the second-year course when calculus is introduced. So, being able to point back to the ski slopes might be a gain in the long run, as students perhaps could be able to recreate the discussions about the good ski slope without cracks and vertical ski jumps.

Ringsted Campus Festival

Optimization of a rectangle area using a quadratic function

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Identifying the problem and learning goals

When students encounter optimization problems in connection with differential calculus, there are two aspects to deal with – setting up a mathematical model that should be optimized and applying differential calculus. Therefore, we wanted to investigate whether, by placing the first part of an optimization problem between a course on polynomials and differential calculus, we could make optimization problems more accessible. The hypothesis was that setting up the functional expression is the most difficult part because the students, once they have the expression, can easily use their tools, regardless of whether it is differential calculus or the determination of the minimum of a quadratic function. Hence the purpose of the lesson was to support the students in setting up a function, getting an understanding of where it comes from, and then optimising using appropriate tools.

In addition, related to the inquiry-based mathematics teaching based on the TDS structure, we wanted to give more time to the validation phase to explore the potential in this phase.

Planning and creating the lesson plan

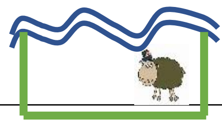
In planning the lesson the team quickly agreed that the solution to an optimization problem should end with an expression of the form $f(x) = ax^2 + bx + c$.

This is to ensure that the students do not use only graphic solutions as an approach to solving the problem and partly to make the process of setting up the functional expression more approachable.

The classic sheepfold task was quickly chosen, but the context was discussed as to whether it makes sense for the students. Would it be better if it was a sandbox up against a wall? A horse fold? The cross-sectional area of a gutter? Here the idea of a festival site arose, where the stage area corresponds to the river and the festival site itself to the rectangular area.

Sheepfold task:

A rectangular fold of sheep is to be built with one side lying against a stream. The length of the fence is 200 m. How wide and how long must the sheepfold be to achieve the largest possible area?



Instead of a 200 m fence, the length of 205 m was chosen to push the students to more easily see the connection between the choice of length and width in relation to the perimeter, i.e.,

$$perimeter = 2 \cdot width + length,$$

must be used to set up an expression for the area.

The problem that the students were given was:

After the pandemic, "Young Ringsted" has decided to hold a one-day festival for young people. They hire the company "Cheap Parties" to make a festival area. The company visits all youth institutions in the area to borrow fences. They manage to collect a total of 205 m of the fence. The fence must surround the festival area on three sides, as the fourth side must be made up of the stage area. How should the festival site be made so that as many people as possible can participate if each participant must have a space corresponding to 1 m²?

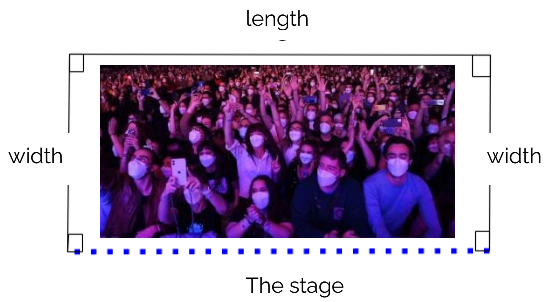


Figure 1. A sketch of the festival site.

The students had a computer, paper, a pencil, and a ruler as well as a piece of string of 205 cm available. It was discussed whether the string was necessary, but the team decided that because some students learn best with something concrete in their hands, then it should be offered as an option.

The 60-minute lesson was planned with only one devolution phase to give students time to work on the problem, and to have time to investigate the validation phase, as the potential in this phase was still unclear to the team. Likewise, before the formulation phase, the students were given 5 minutes to write their solution on an A3 poster, so they did not start thinking about what should be presented from the beginning.

Observation of the Study Lesson

The team has concluded that for the students it is challenging to produce a functional expression from the context. This conclusion was still valid even after two more trials of this lesson. It became very clear after the first attempt that it is important to remove as much noise from the problem as possible. For example, the information that each festival participant had 1 m² caused confusion and additional questions leading the students away from the goal, so this was quickly removed at the next trial. Also, we concluded that to get a validation phase to play a prominent role in the lesson, it is important to have several different answers to play against each other. This did not happen the first time.

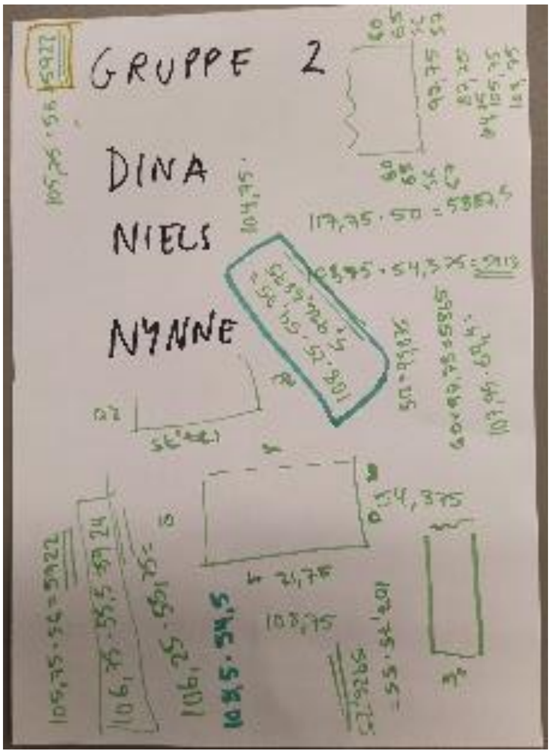


Figure 2. Student work with trials and errors.

The groups that reached the farthest only had a suggestion for the length and width of the festival area based on the trial-and-error method, and their intuition was that there was a "tipping point", because, after a certain width, the area becomes smaller and smaller. The groups that did not reach the target knowledge at all had found three different sizes of the festival area, but not based on calculations, but based on how it would be easiest to find friends at the festival, or based on the best music experience depending on the width of the stage area. So, in the validation phase, there was only one approach for the students to evaluate, i.e., the validation phase turned into a classical lesson conducted by the teacher.

In relation to the use of the strings, it helped students to understand that the area changes when we change the length over width ratio of the rectangle. We had expected to see some students through their trial-and-error method to insert various attempts into a spreadsheet and, using regression, come up with a functional expression. To help students see that they could easily produce numbers to make a data set, an extra devolution phase was introduced, in which students were asked to find the length and width and the corresponding area of a rectangle if the total length of the fence was fixed at 205 m.

This trial of the lesson took the students a step further, but many chose 100 m as the length, and then there were 52,5 m left for the width. The problem is that this is very close to the optimal solution, and became too obvious for the students to choose, so in terms of getting the students closer to the target knowledge, we did not get much further. On the other hand, the validation phase after the first devolution, action, and formulation was examined and here it became clear to the students that not all solutions could be used, because of the connection between the length, the width, and the perimeter. This meant that the institutionalization after this round was that the length could be written as $length = perimeter - 2 \cdot width$.

At the second trial, we had a student who is very skilled at abstract thinking, and she reached the expression

$$Area = width \cdot length$$

$$= width \cdot (perimeter - 2 \cdot width)$$

using this context. This expression was entered into *Ti-Nspire* and the graphical tool *Maximum* was used to determine the largest area. The others in the class were nowhere near this, and when this student had to present her solution, the class ended up in a teaching situation.

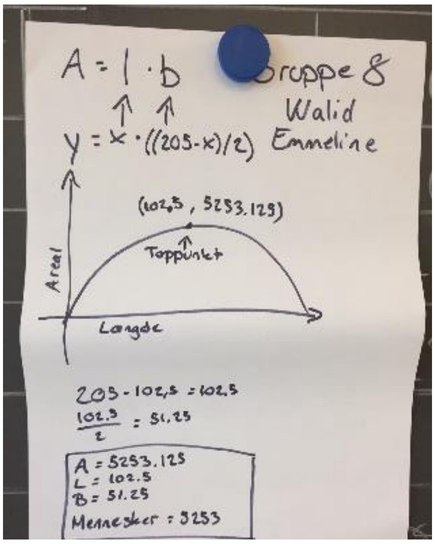


Figure 3. Student's algebraic solution from the second trial

This was similar to the first trial of the lesson, but here it was a student who came up with the arguments as opposed to the first time, where it was the teacher. For the rest of the class, the result was probably pretty much the same - the goal of the lesson had not quite been reached by the students themselves.

In the reflection, we concluded to reformulate the task so that it was strongly emphasized that it was not enough to come up with an answer, but that one also had to reason why they think it is a good answer. It had the effect that it upgraded the trial-and-error solutions, as a group of students argued that one started by dividing the 205 m by 4, as a square would be optimal, while one quarter was then laid lengthwise, so that it became twice a width. This reasoning could be examined in more detail when arriving at the final solution, so in the validation situation, one could argue with this approach.

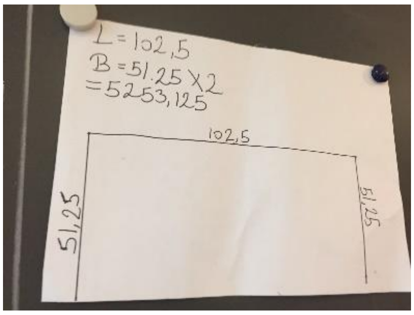
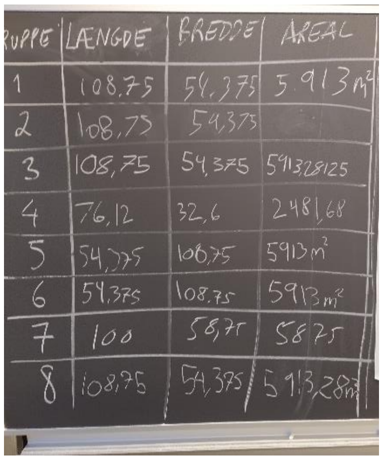


Figure 4. Example of an optimal answer.

An opportunity to do a third trial arose, and here the team retained the strings, the two rounds of devolution, the clarification that argumentation for the solution was important, and finally the removal of 1 m² per participant. To get the students even closer to being able to work with a data set, the length of the fence was changed from 205 m to 217 m. This was to avoid the students' obvious choices.

In this trial, the validation phase after the first devolution was again successful, as the students realized that the connection between perimeter, length and width has to hold. None of the students set up a function expression algebraically, as seen in the second trial, but in return, a group of the students chose to systematically test all lengths from 1 to 217 using the CAS tool, and using regression they reached a second-degree polynomial, whose extrema could then be looked for.



RUPPE	LÆNGDE	BREDDE	AREAL
1	108,75	54,375	5.913m²
2	108,75	54,375	
3	108,75	54,375	5913,28125
4	76,12	32,6	2481,68
5	54,375	108,75	5913m²
6	54,375	108,75	5913m²
7	100	58,75	58,75
8	108,75	54,375	5913,28m²

Figure 5. Table with different answers from the third trial.

On the other hand, for most students, it was still trial-and-error and satisfaction once they had found a plausible answer. In the second trial, all students chose the same number to start with, but in the third trial, the table on the board contained different possible dimensions including those for which the area is the largest. This did not invite the students to try other options further, and they did not have an inner need to see a general solution. So, it was the argumentation of their own solution that became the point for the students in the third trial.

Reflection

The scenario in its current form could be interesting to be tried out again, but with the attention on that the first devolution should invite the students to a higher extent to choose different values for length and width, so it becomes clear that we are faced with a data set. Could one expect the students to make the regression and then discuss why should it just give a parabola as a graph?

One might also be tempted to ask whether it is possible for students to make an algebraic expression at all. We discussed this in-depth after the third trial because one of the weaknesses of Danish students is setting up abstract expressions in general. The conclusion was that the most difficult part of the optimization problems is not the optimization itself, but creating the functional expression that must be optimized.

Perhaps, if the students met these kinds of problems earlier in their math education, it might be possible to recall this knowledge when it is needed before calculus. It could be very interesting to practice setting up expressions from a text with first-year students to see if it would help them with the optimization problems.

The number game

Why the square root is not a negative number

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Defining the problem and target goals

In this study, we have dealt with a subject that our students often have trouble understanding. Students are confused about why a square root of a positive real number is a unique positive number. Lack of clarity arises from the fact that when squaring a positive number and its opposite negative number the result is the same. The fact that the root of zero is uniquely determined (i.e., $\sqrt{0} = 0$) adds to the confusion.

Students know that squaring a real number can never result in a negative number, i.e., we will not be able to find a square root of a negative number inside the set of real numbers $\mathbb{R}$, but this is a fact which is somehow unrelated to the fact that a square root cannot be negative. So, the students often have doubts why $\sqrt{x^2} = \pm x$ is not a correct answer, although $(-x)^2 = x^2$. More specifically, why we cannot say that $\sqrt{9} = 3$ and $\sqrt{9} = -3$ when both 3 and -3 squared give 9, i.e., $(-3)^2 = 3^2 = 9$.

The target knowledge for this lesson is the difference between the square root of a non-negative number a and the solutions to the quadratic equation $x^2 = a,\ a \in \mathbb{R},\ a \geq 0$. Broader goals are communication, the square root function, and the quadratic equation in the form $x^2 = a,\ a \in \mathbb{R},\ a \geq 0$.

The lesson consists of solving simple mathematical tasks and playing a board game named "The number game". Players move their figures clockwise or anti-clockwise (depending on the solution of the task) on the board and then compare which field on the board was the final field for their meeple. They make conclusions about the ambiguity of solutions for some of the given tasks.

Planning and creating a lesson

The duration of the lesson is two school periods (80 minutes). Students work in groups, 4-5 students per group. The lesson is organized into four phases. Students are expected to solve mathematical tasks, participate in discussions, and compare, define, and present their conclusions. They will play "The number game" where they will need to solve tasks such as "Square root of 25" and "Number whose square is 9". Students who opt for a negative number when solving a square root will end the game on a field that contains negative points.

Phase one: "The number game" – the instructions and rules for playing the game.

Material is handed out to students: game board for "The number game", cards with printed tasks, and rules of the game. The youngest player starts the game, pulls out a card and together with the rest of the group members solves the task. Figure is then moved, clockwise or anticlockwise, by the number of fields depending on the result of the task. After the game is played the students mark the field where the game ended for their group.

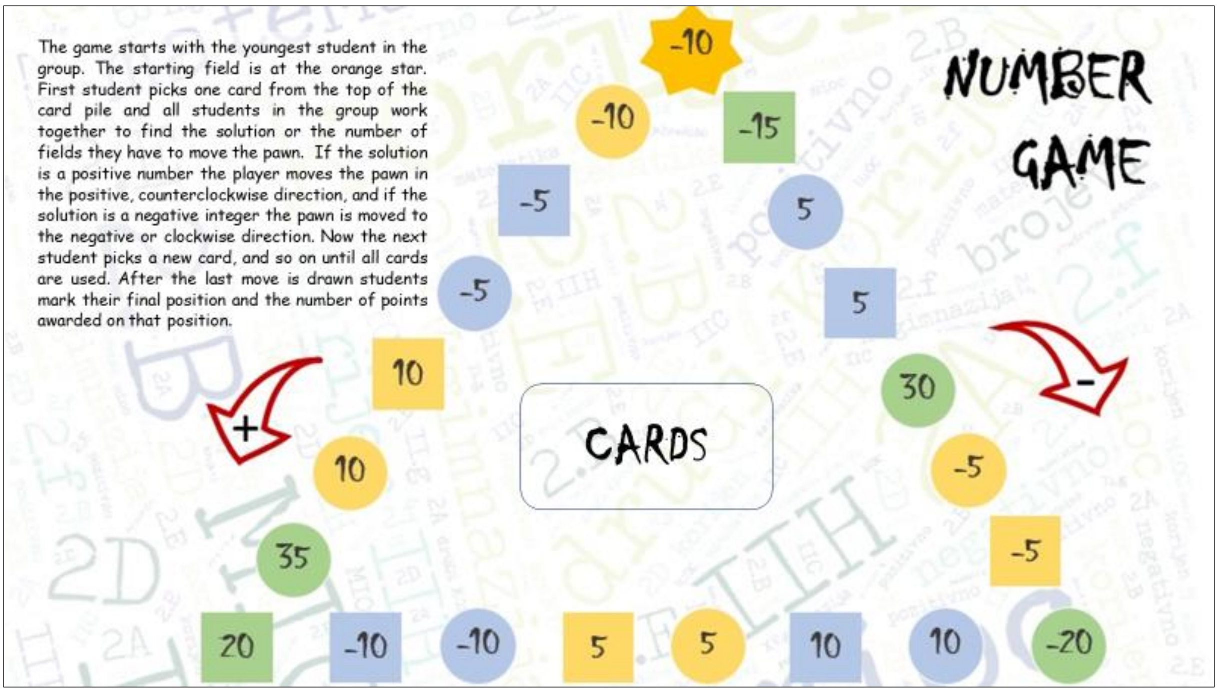

Figure 1. The game board

A. The sum of all solutions $(x+7)(x-6)(x+5)=0$	**B.** A positive divisor of 2	**C.** The smallest of three consecutive integers whose sum is -15	**D.** The square root of 25
E. An integer solution of the equation $3x(2x-5)=0$	**F.** The difference of the squares of the numbers 4 and 3	**G.** The cube of the sum of the numbers 1 and 3 divided by 8	**H.** A number whose absolute value is 4
I. A number whose square is 9	**J.** The arithmetic mean of the Pythagorean triple $(5, 12, ?)$	**K.** A number opposite to the number of diagonals of a convex pentagon	**L.** The median of the given numbers: $-\frac{7}{2}, -11.5, 0, -\sqrt{3}, -3$

Figure 2. Cards with the printed tasks

Second phase: presenting the results

Students present how many points they won and the position where the game ended. The teacher marks that on the board. They realize that even though they all got the cards with the same tasks and the same game board they did not necessarily end the game in the same position. They conclude that some tasks do not have a unique solution.

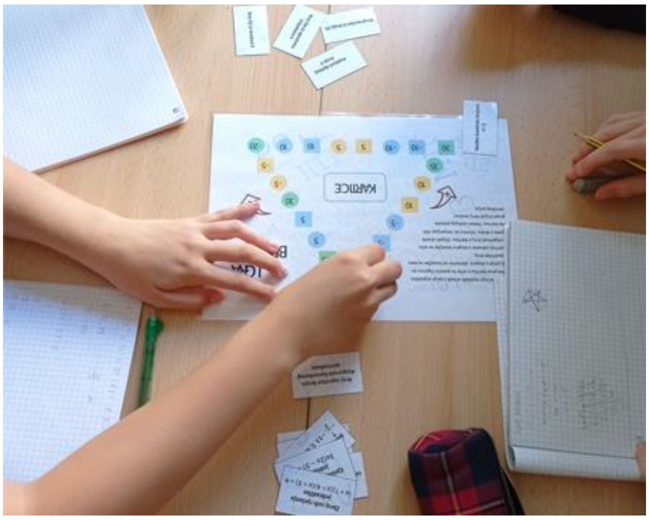

Figure 6. Sorting the cards

Third phase: analysing the game results

Students single out cards that contain tasks which do not have unique solutions and analyse (drawing trees, listing sequences, sorting the data using tables…) all the possibilities of how the game can end. They make inferences about the different number of points on fields, e.g., are all outcomes equally good and if not, why so?

Phase four: mathematization, presentation, institutionalization

Students describe cards that have two solutions and focus on the two most significant ones: “Square root of 25” and “Number whose square is 9”. They discuss whether both tasks have two solutions and why.

The teacher gives an answer using a specific example and then presents the historic background for the introduced term, *the square root*. Afterwards, the students attempt to write the definition of the square root and the solutions of the quadratic equation of the form $x^2 = a,\ a \in \mathbb{R},\ a \geq 0$. Groups write their definitions on papers which are then presented and compared with other groups. They discuss whether all definitions are correct and mathematically precise. The lesson ends with the teacher writing the definitions on the board.

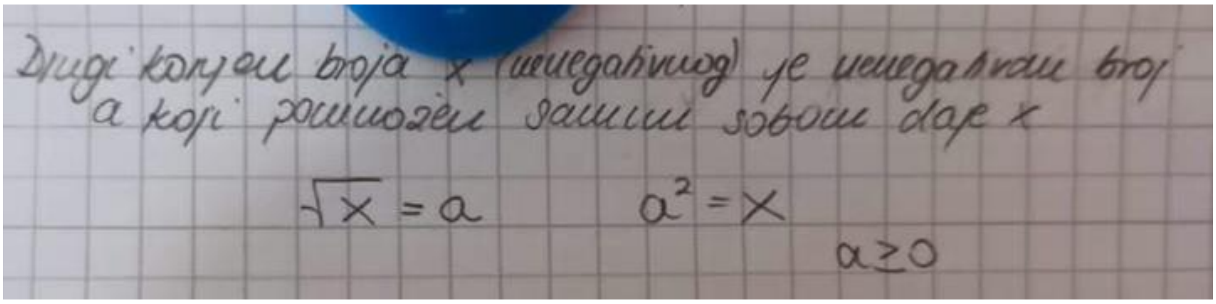

Figure 4. A student's definition

Observation

At the beginning of the lesson, the teacher handed out the materials and explained the rules of the game to the students. Students quickly discovered that some cards have more than one solution and are not sure what to do in this case. Their opinions differed. We noticed they were trying to develop a strategy for the game, such as: “*It is always better to move in the same direction and accept only positive solutions.*” or “*This time it is better to move in the opposite direction because we will end up on a field with more points.*” or similar. When the cards with a unique solution were singled out, some groups concluded that the outcome is not dependent on those cards, but rather on the cards that have two solutions. Some groups singled out three and some four such cards. Some groups were leading a discussion on where to put the card with the task “Square root of

25", i.e., does this task have a unique solution 5 or is the number -5 also a solution. They would most often accept the suggested solution from a calculator that surely "knows the best". Then they wrote out all the possibilities of how the game can end. During the game analysis, it has been discovered that some groups ended the game on a field where they were not supposed to. That was a sign for the teacher that a mistake has been made – either while solving tasks or while counting the fields and moving across the board.

The teacher then told them the story about the history of the square root and gave them some examples showing the difference between the square root and the solutions of the quadratic equation of the form $x^2 = a,\ a \in \mathbb{R},\ a \geq 0$. Now, the students had to write the definitions in groups.

When the discussion led by the teacher was done, the students briefly commented on the accuracy and precision of their definitions. Both definitions were then provided (written) on the board by the teacher at the end of the lesson.

Reflection

The lesson was held four times and based on the observation minor changes were made in the scenario. This lesson was structured and led by the teacher, but the students were engaged during the entire lesson: while playing the game, discussing, presenting conclusions, and formulating the definition of the square root and the solutions of the quadratic equation in the form $x^2 = a,\ a \in \mathbb{R},\ a \geq 0$.

Two unexpected situations occurred:

- singling out cards with the difference of squares as ambiguous.
- using the term *absolute value* while defining the square root.

Further suggestions based on previous experience:

- allocate more time for discussion about students' definitions.
- convince students to pay closer attention when other groups are presenting their conclusions.

Fruit for snack

Domain restriction needed for the inverse function

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Identifying the problem and learning goals

Finding an inverse function for a given function often becomes a recipe for students and they easily forget to question the existence of an inverse function, especially when simple functions are in question. In the 2nd grade (of Croatian high schools) students learn about the quadratic function and its inverse, the square root function. They quickly grasp that the square root function is the inverse to the quadratic putting aside that this could be obtained only for a specific "part" of the quadratic.

Hence, we have chosen to emphasize the necessity of discussing the domain of a given function to establish requirements for the existence of the inverse function. The target knowledge for the Study Lesson is the domain restriction, with broader objectives: communication, establishing requirements to construct an inverse function, and making a given function injective and surjective. Since we have decided to focus on domain restriction exclusively, we have chosen examples where the codomain is the image of a given function (in other words, we were observing surjections).

The lesson is planned in a non-mathematical, real-life context to motivate students to derive mathematical restrictions based on the necessity to restrict a real-life situation. The main idea is to observe „*wishes*" (desired fruits) vs. "*granting wishes*" (distribution of fruits).

Planning and creating the lesson plan

The lesson is planned for 80 minutes and it was performed in an online teaching/learning environment. The lesson is organized in four 10 minutes group-work sessions, in which students, based on the given material, were expected to discuss, compare, define, and communicate their findings. The material prepared for the lesson includes four materials handed to students at different stages of the lesson.

The students are first given the "set-up": a problem of a family with five children, where each day each sister asks for one piece of fruit. On the other hand, for each day of the week, there are different fruits offered.

	Danijela	**Jelena**	**Maja**	**Sanja**	**Vita**
Monday	Banana	Apple	Pear	Orange	Plum
Tuesday	-	Apple	Pear	Orange	Plum
Wednesday	Banana	Banana	Pear	Orange	Plum
Thursday	Banana	Apple	Pear	Orange	Plum and pear
Friday	Apple	Apple	Orange	Orange	Plum
Saturday	Banana	Banana	Banana	Banana	Banana
Sunday	Apple	Banana	Plum	Orange	Pear

Table 1. Wishes for each day of the week

Phase 1. Students discuss if there are days of the week when sisters do not properly choose a fruit (like Danijela wishes no fruit on Tuesday, etc.) By finding such days, students exclude those that are not representing functions (though we do not use that language yet). We call these mappings from sisters to the provided fruit "wishes".

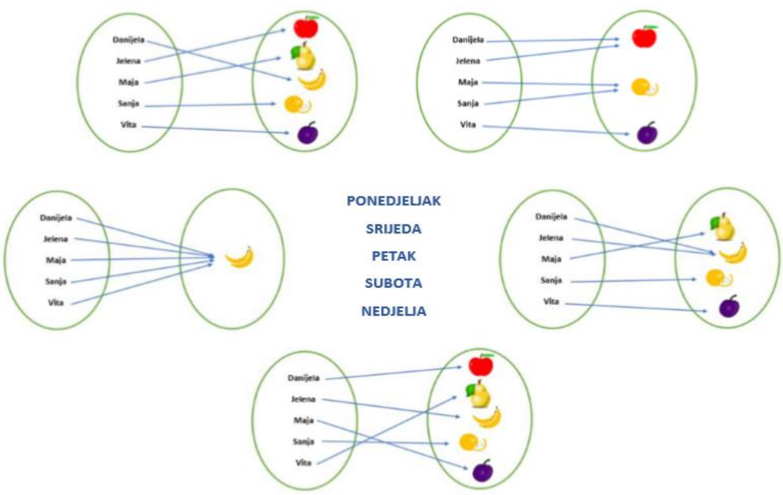

Figure 2. Diagrams representing wishes

Phase 2. Students pair diagrams with the stories (days of the week), recognizing and subconsciously comparing "wishes" from the story to the concept of a function. To confirm this, the teacher organizes a plenary discussion of when and why girls could have their chosen fruit.

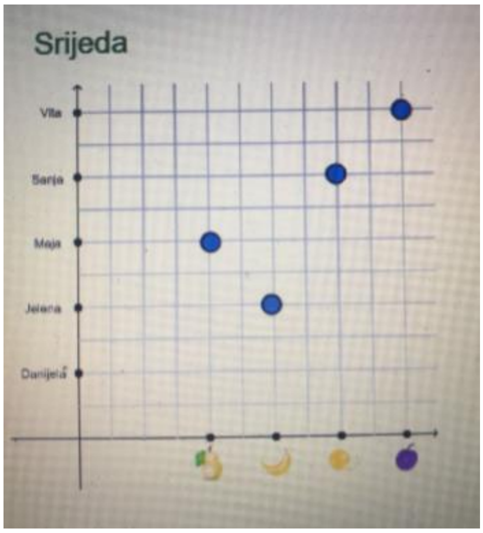

Figure 3. A discrete graph of a possible distribution for Wednesday

Phase 3. In the groups, students are given labelled coordinate systems where they graph "wishes" and discuss further if the fruit can be distributed by their whish or do they have to alter sisters' wishes, and if "Yes", how to do that? In each group, the teacher selects one "mother" and the group consisting of mothers distributes fruit each day, also representing such a distribution as a discrete graph. Again, the teacher leads a plenary discussion of "if, when, and how" the fruit can be distributed respecting the sisters' wishes.

Phase 4. Students compare and communicate the concepts of "wishes" and "distributions" using the mathematical language of functions and their inverse functions. To support them in formulating mathematical conclusions students are offered "matching terms" material as a *Wordwall* activity.

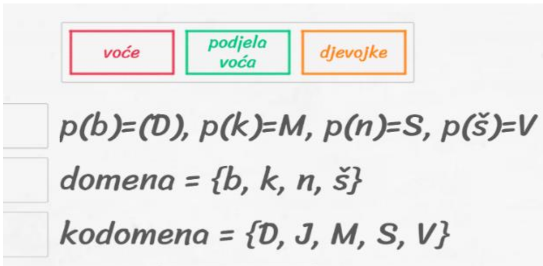

Figure 4. Wordwall activity

The goal of the activity is to reduce the set of sisters on certain days! Students present their group formulations on a Padlet, based on which the teacher synthesizes the existence of an inverse function given that the original function was surjective.

Observation of the Study Lesson

In the first devolution, the teacher presents a real-life situation to the students and they need to recognize where the problem of incorrect "fruit choice" is present. For most students, this was quickly and logically done. Students excluded Tuesday and Thursday explaining their reasoning by: "*She didn't choose any fruit*" or "*She has chosen two different* *fruits*". None of this was allowed by the problem set-up. Students were also presented with the diagrams and they quickly paired the diagrams with the days of the week.

In the next devolution the students were supposed to graph "wishes", and one group (the "mothers") discussed and graphed the "distribution" of the fruit. "Mothers" had difficulties distributing fruit objectively, they incorporated their emotions and unfair feelings into the distribution. Hence it took them longer to decide about the final fruit distribution.

In the third devolution, students were asked to describe and communicate "wish C" and "distribution D" using mathematical language. Their formulations were posted on Padlet. Students used words like the domain, the codomain, the switch of the domain and the codomain regarding C vs. D, reducing the domain for the "wishes" to be able to produce a "distribution". Here is one example of students' formulation:

> *At first, we had "wishes" and that was the assigning rule. The set of sisters was the domain and the set of fruits a codomain. When we were supposed to map fruits to sisters, the rule of a function did not hold. So, we have reduced the set of sisters and now every element in the domain has only one element assigned from the codomain.*

In a guided discussion, engaging all students, the teacher addressed all terms and translated every mathematical word into a corresponding symbol. When needed the teacher introduced new terms: the inverse function and the domain restriction.

Reflection and concluding remarks

The lesson was very structured and guided by the teacher. Even though students discussed and presented their formulations there were four instances during the period where the teacher summarized their ideas and formulations before continuing.

An unexpected moment occurred when "mothers" could not distribute the fruit because they got tangled in a life-like situation rather than objective mathematical thinking.

Students successfully described targeted concepts of a function, the inverse function, and the domain restriction, but had difficulties using symbolic mathematical language to write those down.

Alterations for future lessons based on the experience:

- Plan a more unstructured and less guided lesson,
- Introduce more groups of "mothers" instead of "sisters", so that more students discuss "distributions",
- Adapt the learning goals to introduce the definition of a function,
- Emphasize two different mappings "wishes" and "distributions" and their inverse relationship.

Volumetric flow rate

Modelling a real-life situation with the square root function

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Defining the problem and target goals

When discussing expected learning goals in the national curriculum, the student is required to recognize the problem and model the situation using elementary functions. We can say that the students are doing well when faced with modelling situations that can be resolved with the use of linear functions. When faced with quadratic functions and square root functions we often notice some difficulties.

Mathematical modelling can be described as a process whose purpose is to describe a real problem by applying mathematical reasoning and concepts. We start with a real-life situation and from there try to obtain the right mathematical model.

The connection of the observed problem with a mathematical object is difficult on its own, which is why we focused our lesson study on the importance of modelling. To do so, we used a real-life situation which will be modelled by a square root function.

The target knowledge for this unit is the square root dependence of height over time (by observing water levels in a prism-shaped bowl, that are rising at constant speed). The broader goals of the lesson are calculating with different measuring units, data organization, data analysis, determining the expression/rule which defines the relationship between the variables, sketching the graph of the square root function, the relation between the quadratic and square root function, domain restriction, connecting mathematical reasoning, communication and methods combined with physics.

Planning and creating a lesson

This study lesson is planned for two school periods (90 minutes), but was held during two periods that lasted 80 minutes. It is organized into four phases. The students are paired up during each of these phases. Before each phase, the students are provided with the necessary materials (video for phases 1 and 3, and foils for phase 2).

The students were presented with the following problem: In the video, you will see water being poured into a prism-shaped container. Watch the video (link: https://bit.ly/watervideopart1) and describe how the level (height) of the water changes over time. Justify your answer the best you can using mathematical reasoning and notation (graphs, tables, and formulas).

Phase one: analysing the problem and finding the dependence

The students are watching the first part of the video after which they attempt to describe how the water level has changed over time. The students are looking for the relationship between the height of the water level and time, using different methods (deducing the formula, pausing the video, measuring and plotting data in a coordinate system).

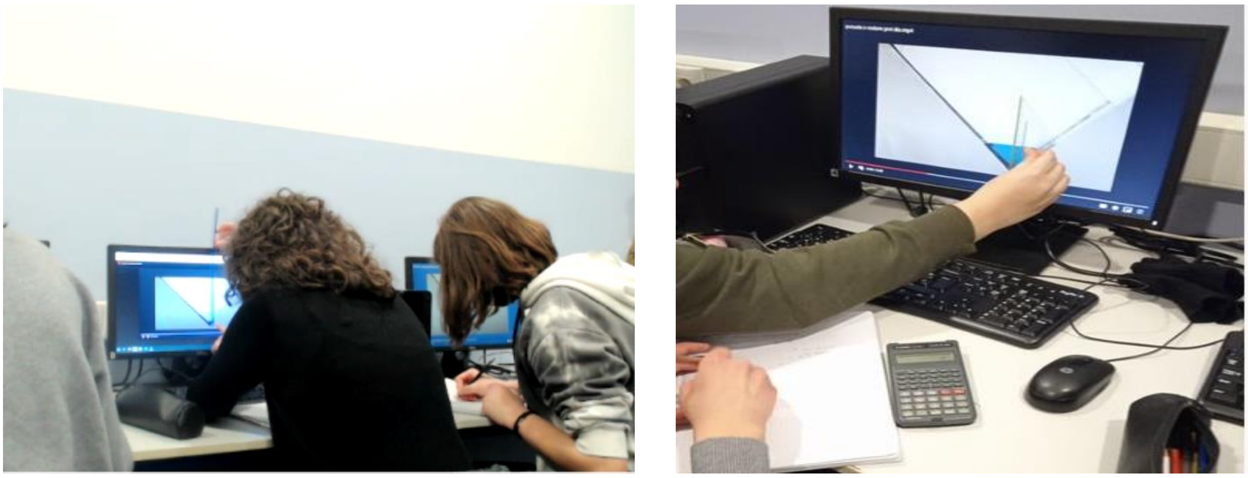

Figure 1. Students watching the video and measuring the height of the water

Phase two: preparing presentations

After the first phase, which lasted around 40 minutes, during the second phase, students needed to write their conclusions on foils (provided by the teacher) and present their ideas.

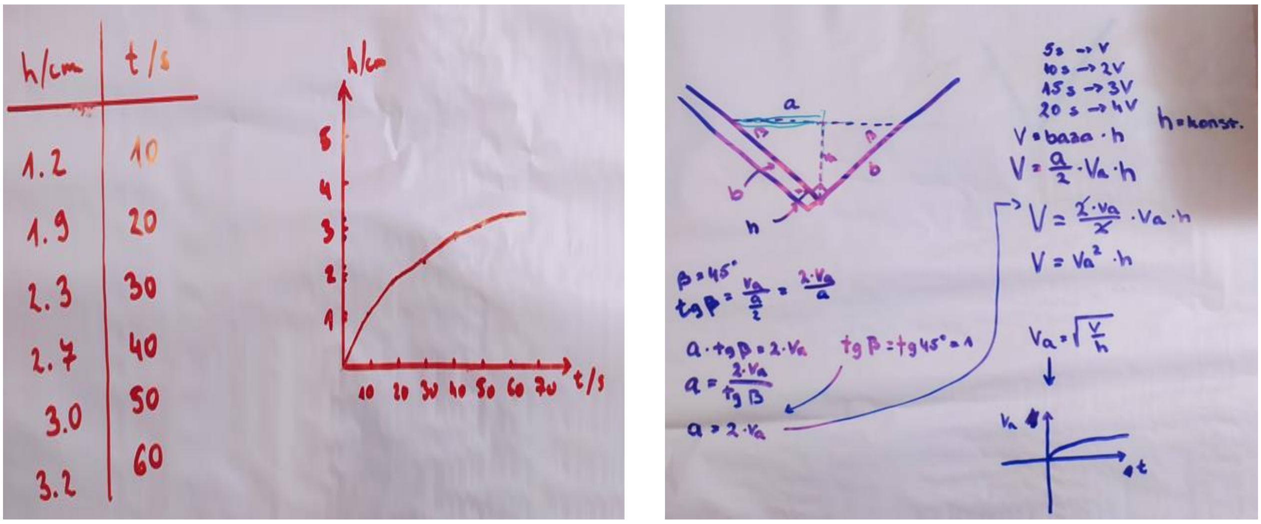

Figure 2. Derivation of the formula for the dependency (left) and the graph of the regression function (right)

Phase three: estimation

Based on their work students assess the height of the water levels after a certain time (a couple of minutes) and write their assessment on the foil. During this phase, the students will see the second part of the video and verify their assessments. Since there were no groups that obtained the formula-based rule (function dependence), the teacher decided not to show the second part of the video. Only a few groups were able to make and write down their assessments.

Phase four: presentations and institutionalization

The students place foils on the board and describe their conclusions while listening to others present their work and participating in a teacher-led discussion. Based on the students' work, the teacher encourages students to make inferences about the dependence of the level of water in a container over time. If the container is filling with water at a constant speed, then this dependence is a square root dependence. In the end, they define the square root function, and the students write the definition in their notebooks.

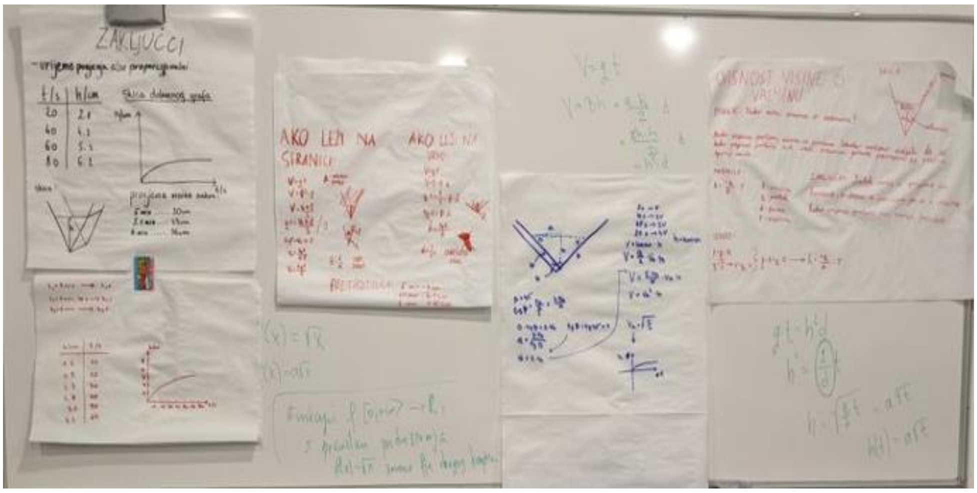

Figure 4. White board with students' presentations

Observation of the Study Lesson

During the first phase, the teacher presented to the students a real-life problem that needs to be described using a mathematical model. Most of the students did not know how to approach this problem. The idea of measuring and using a ruler on a screen did not occur to them naturally, although some of them tried it later. Some of the students tried to obtain the formula directly.

Some groups assumed that the height is proportional to time. As time passed, they noticed that this was not true but still made several validations expecting their assumptions were right. The group that tried to obtain the formula directly made some errors while calculating and maintained the conclusion that the two observed values are proportional. In the end, most of the students realized that the dependence in question was a square root dependence but still no one found the right formula.

In the second phase, students wrote their conclusions on foils although they were aware that the work was not complete. This was the reason why the plan for the lesson was changed and the students did not receive the second part of the video, whose purpose was to validate students' assumptions. Just a few groups were able to make estimations based on their work.

In the final phase, each group presented their work on the board. They briefly explained it and shared their approach to the problem with the rest of the groups, mentioning all the obstacles they met during their research and being aware that they have not reached the final goal. However, each group had something that was leading the research in the right direction. Together with the teacher, during the led discussion, by combining the work of all groups, the dependence of the level of water over time was determined and the goal of this study lesson was achieved.

Reflection

Lesson study was planned so that all group work is independent. All students worked hard, put in a lot of effort and made some good conclusions but none of the groups obtained the right formula on their own. Although we are aware of the high demands that were placed upon our students, we might say we were not expecting this outcome. We expected our students to measure and show their data in the coordinate system, but they were very reluctant when it came to this idea.

Nevertheless, the target goal of the lesson study was achieved in the end and the students quickly recognized what was missing and how it could have been obtained. Considering how demanding the problem of the volumetric flow rate is, a possible change that can be made to this lesson study is to use a geometrical problem instead. This problem could be presented in the following manner:

> *In this video, you will observe the right-angle triangle whose altitude on the hypotenuse and one of the segments of the hypotenuse are changing. Describe how the height depends on the length of the segment. Video:* https://bit.ly/trianglevideo1

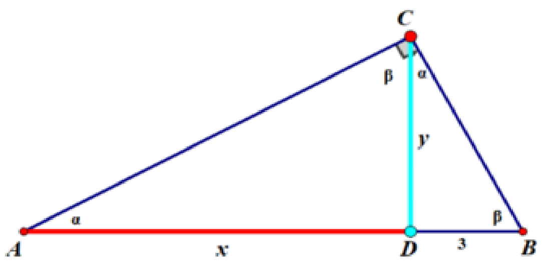

Figure 5. Sketch of the situation in the follow-up problem.

The logarithmic scale

The difference between the linear and the logarithmic scale

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Identifying the problem and learning goals

The trigger for identifying this problem was the earthquake in Zagreb on the 22nd of March 2020 of the Richter magnitude 5.5. After the main shock, there came many aftershocks with varying magnitudes and people were making the same fuss if they heard that there was an earthquake of magnitude 4 as for those of magnitude 1.8. It came out that they do not know that if a magnitude is less by one it means that the earthquake is about 32 times weaker, if a magnitude is less by two we have about a 1000 times weaker earthquake, and so on. The reason for this is that the formula for calculating the Richter magnitude involves logarithms: $M = \frac{2}{3}(log\ E - 4.8)$. We can see it is the logarithmic scale, and not the linear, to which we are most accustomed.

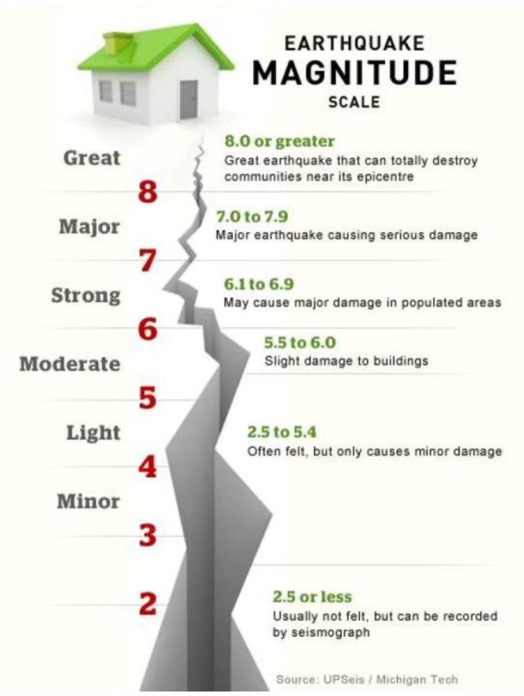


Figure 1. Richter earthquake magnitude

Another contemporary topic is the Covid-19 pandemic and in the media one can find all kinds of data represented on different scales which can be misleading if you don't pay attention. The exponential growth seen on the linear scale diagram does not look so bad if you put the same data on the logarithmic scale diagram. We wanted to make sure that the students are aware of how important it is to look at the labels on the axes in a graphical representation of data before making conclusions.

Since the exponential and the logarithmic function are highlighted in the curriculum for the third grade of high school, we decided to address the problem of linear thinking in a nonlinear world as an introduction to this theme and give the logarithmic function another purpose than the inverse function of an exponential function. The target knowledge for the Study Lesson is: Applying logarithmic scale, with broader objectives: Understanding various graphical representations of the same data, Choosing appropriate graphical representation and Real-world problem-solving.

Planning and creating the lesson plan

The lesson is planned for 80 - 90 minutes. The class is divided into eight groups of three (ideally) and each group is given a table with raw data (two groups work on the same data) and an A3 blank piece of paper. The students' task is to represent the data graphically in a coordinate system.

We wanted to use real data and our long research brought us to these sources:

- for Covid: web page Italy Coronavirus;
- for kefir: article Observation of Lactic Acid Bacteria and Yeast Populations During Fermentation and Cold Storage in Cow's, Ewe's and Goat's Milk Kefirs;
- for noise: Institute of Safety Research and Development;
- for earthquakes: web page LastQuake.

After this phase (about 30 minutes), one student from each group comes in front of the class to present their work only stating what data they had, what problems they encountered and how they solved them. The teacher doesn't comment on their work.

All diagrams are left on the board pinned with magnets so that the students can clearly see how the same data can have different graphical representations.

The table provides data on the total number of those infected in Italy during quarantine in the spring of 2020.

Date (2020.)	Total number of infected
17.2.	3
24.2.	229
2.3.	2038
9.3.	9179
16.3.	27997
23.3.	63941
30.3.	101723
9.4.	143612
16.4.	168932
23.4.	189957

Graph this data in a coordinate system.

The table provides data on the concentration of hydrogen ions during the fermentation of kefir added to milk.

Elapsed time (in hours)	Concentration of hydrogen ions (mol/L)
0	0.000000095
4	0.000000240
8	0.000000437
12	0.000000912
16	0.000001995
20	0.000015488
24	0.000028840
28	0.000114815
32	0.000354813
36	0.001584893

Graph this data in a coordinate system.

The table provides data on the noise level during the 20 minutes of the concert held in July 2020.

Time (9.7.2020.)	Noise level (μPa)
22:17	2907.56
22:19	493776.02
22:21	275124.96
22:23	6645.53
22:25	911024.37
22:27	643472.50
22:29	101281.46
22:31	1075300.86
22:33	1782501.87
22:35	23228.97

Graph this data in a coordinate system.

The table provides data on the amount of energy released in the earthquakes in Zagreb between 22.3 2020. and 26.3.2020.

Earthquake number	Released energy (J)
1.	7943300000000
2.	1995300000000
3.	22387002541
4.	5623400000
5.	1000000000
6.	1412500000
7.	1995300000
8.	22387591200
9.	3981100000
10.	31623000

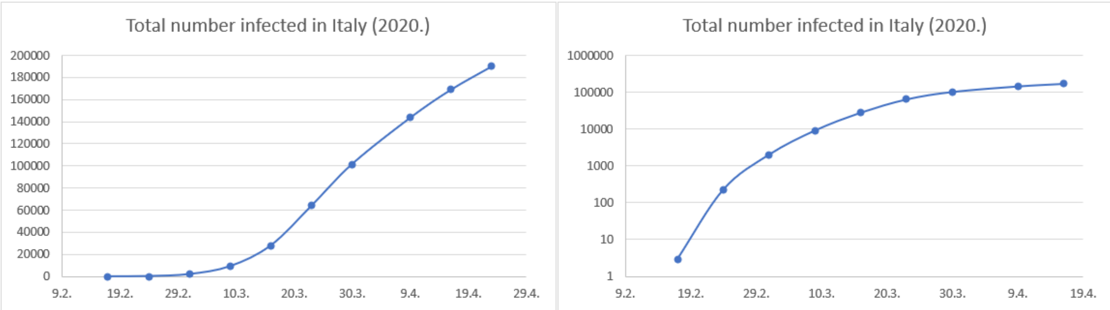
Graph this data in a coordinate system.

Figure 2. Raw data tables for the groups

In the second phase, each group is given a handout with two graphical representations of the data they were working on (one has y-axis values on a linear and the other on a logarithmic scale) and a set of three questions from which they need to decide which representation is better. They were given instructions to prepare a short oral presentation for the rest of the class in which they will present: *What can you deduce from the given graphs? Which conclusions are easier to draw from the left and which from the right graph? Which graph offers you better information?*

We prepared a PowerPoint presentation with all the diagrams so the students giving an oral presentation would have it to better explain their topic and argue their conclusions. This is important so all the students can see all the diagrams and participate in the discussion.

COVID-19 IN ITALY



Answer the following three questions as a group.

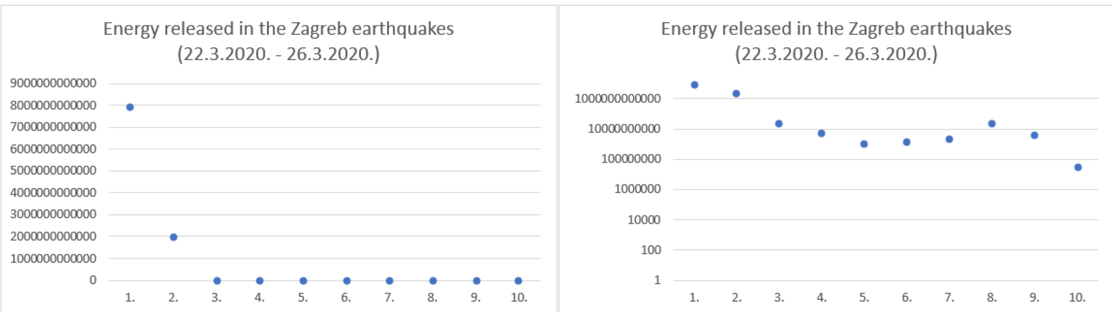
1. Estimate the number of infected on 30.4.2020.

2. Between which dates did the biggest change in the number of patients occur?

3. At what point does the growth of the number of patients slow down?

Figure 3. Handout for Covid groups

ZAGREB EARTHQUAKES



Answer the following three questions as a group.

1. Was more energy released in the 5th earthquake or in the 8th earthquake?

2. Between which two of the earthquakes shown is the largest change in the amount of energy released?

3. Most people will feel earthquakes in which the amount of energy released is greater than 63095730000 J. Which of the earthquakes shown did most Zagreb residents feel?

Figure 4. Handout for earthquake groups

In the final phase, a plenary discussion takes place. The teacher underlines the best conclusions and finishes with a statement "*We think linearly, but our senses are on a logarithmic scale.*" while giving formulas for "taste", "sound" and "movement". This is where they are motivated for knowing all about this *log* that appears in the formulas mentioned. Now they know it is useful and are very curious. Groups that rediscovered the logarithmic scale will feel proud of this success.

$$pH = -\log[H^+]$$

$$dB = 20\log\frac{P}{P_o}$$

$$M = \frac{2}{3}(\log E - 4.8)$$

Figure 5. Examples of logarithmic scales

Observation of the Study Lesson

After the first devolution, students in groups start drawing a coordinate system full of confidence the way they usually do it - the paper is turned landscape, the origin is in the middle of the paper, axes are named *x* and *y*, the unit is the same on both axes, and the scale is linear. Contrary to their expectations, as they look at the data, they encounter some problems. They notice that all the values are positive, so they only need the first quadrant and place the origin in one corner of the paper. Some groups decide that it's better if the paper orientation is portrait. The biggest problem is how to fit everything on the *y*-axis. This is what most of the discussion is about. Some groups decide to scale the *y*-axis values up or down linearly (depending on whether their data is very small as for kefir, or very big like the earthquake magnitude) and some indeed come up with the idea to show powers of 10.

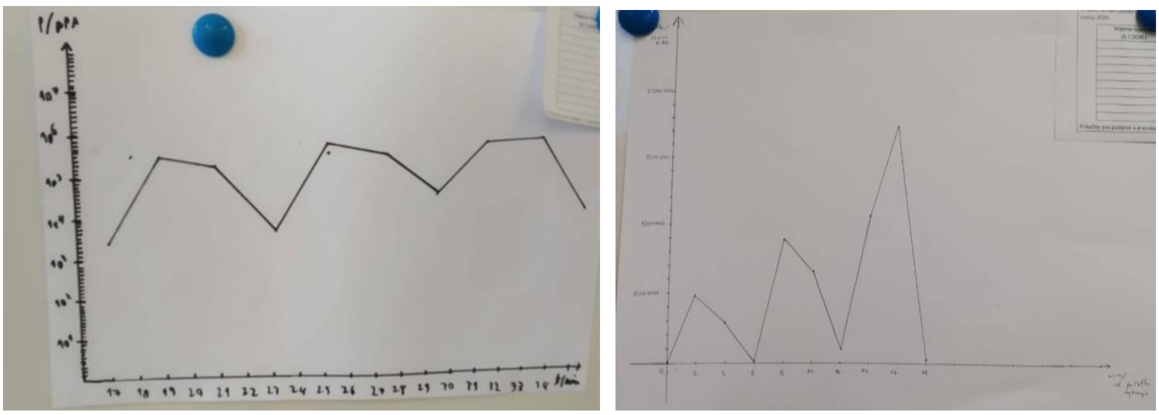


Figure 6. Examples of students' work, Noise groups

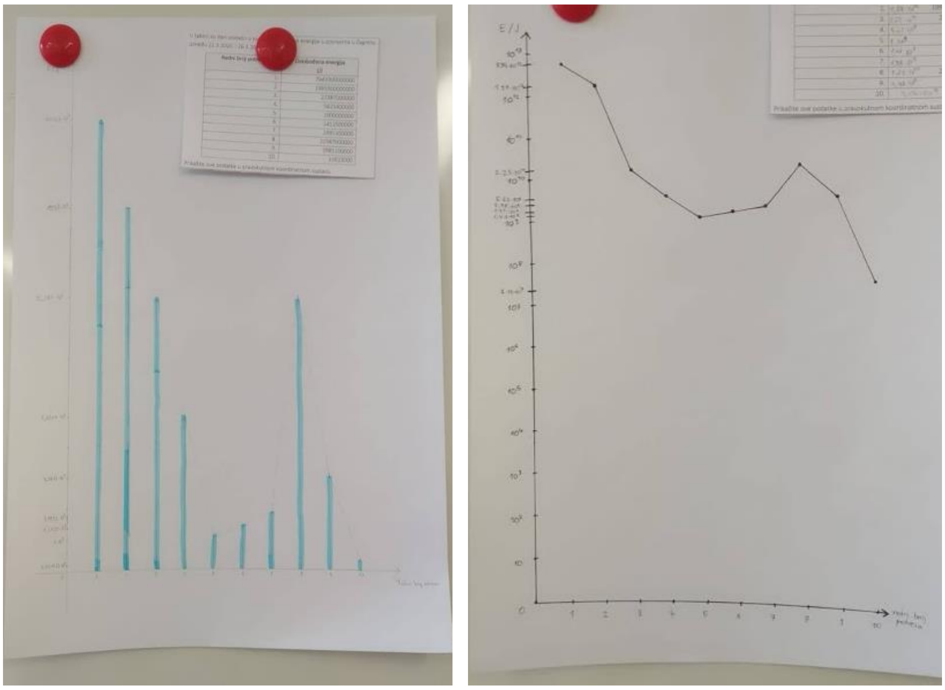


Figure 7. Examples of students' work, Earthquake groups

After the second devolution students eagerly discuss the questions asked on the handout. Some interesting conclusions during the presentation of group work were:

- Both representations are good, depending on the information you seek.
- Linear scale gives us a better picture of the actual data, but it's hard to put all the data in.
- Logarithmic scale solves the problem of putting different scale data on one graph, but it can be misleading.
- On the linear scale, we see the absolute change, on the logarithmic scale we see the rate of change.
- We have to pay great attention to what is written on the axes.

After each of the three times we held this lesson, we took student feedback. Here are some of them:

I learned that I need to pay more attention to the axis of the graphs. I also learned that this is actually useful in real life.

I was surprised at how different graphs with differently arranged values on the y-axis could look.

I was surprised at how manipulated data can be by displaying it graphically.

I learned about the difference between a linear and a logarithmic scale, the "naturalness" of the former, but also the usefulness of the latter.

I have learned that data can be displayed in multiple ways through multiple scales and graphs, and that each method has its benefits in reading some of the required information.

I learned how to draw and read graphs better, pay attention to what is written on the axes, not just look at the numbers, for the first time I saw graphs on a logarithmic scale and compared it with normal ones, had fun...

This is how we knew we hit the target.

Reflection and concluding remarks

The lesson was very successful, students managed to reinvent the logarithmic scale, noticed the importance of careful graph reading and understood the "*real-lifeness"* of logarithms - mathematics has become relevant. In the final phase, the teacher only needed to say "*Yes, you are all right*" because they have said everything that has been set as a target.

At the end of the first phase, it was interesting to see that

- one group has zoomed in on a part of a graph,
- some groups realized and argued that the earthquake graph cannot be shown continuously - it has to be discrete,

- one group noted that one point did not fit on the paper and that it would be 150 cm above it.

In future lessons, it would be better if the diagrams were drawn with felt pens for better visibility - one for each group could be handed out along with an A3 paper and, if possible, a big ruler.

Note that students need at least 30 minutes for the first phase, and 10-15 minutes is enough for them to work on the questions. Be sure to leave enough time for the final presentations, discussion and teacher's formulation. Teachers should make sure students present only what was asked of them and not their entire work. It is good to make students write individual feedback because this way you make them think and put in words what they have learned. And if they put it in words they remember better.

Distance and angle

Constructing models and definitions for basic spatial concepts

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Identifying the problem and learning goals

The motivation for this lesson study was based on our previous experience that students have difficulties in defining basic terms in stereometry, visualizing the geometric problem and applying definitions to problem-solving. We were not pleased with the way this topic is usually introduced in high school textbooks and considered we should give students the opportunity to reveal important definitions by themselves, rather than reproducing given formal definitions.

The target knowledge for this lesson is defining basic terms in stereometry such as distance and angle, understanding written definitions and making an appropriate manipulative model based on a given definition.

Planning and creating the lesson plan

The lesson is planned for 80 minutes. The class is divided in half, and each half into three groups of four (ideally). Each group is given an A3 worksheet leading them to define a term on their own based on a real-life example from immediate surroundings (see Figure 1). One group from each half of the class works on the same term.

We wanted to use examples from their classroom to help students visualize the problem and come up with the appropriate definition.

There were tasks for three different terms, distributed to six groups. The concepts that are studied are: the distance from a point to a plane, the angle between a line and a plane, and the angle between two planes. In each group, there are three questions, the first one focusing on a real-life example, the second inviting to measure the concept in the example, and the third asking the students to define the concept based on this example. For each concept, there were two real-life examples. All the tasks are given in the worksheet attached to the scenario.

What is the distance between the projector lens and the projector screen?

How would you measure that distance?

Based on this example, try to define generally the distance from point to a plane.

Write your definition neatly:

DISTANCE FROM A POINT TO A PLANE

Figure 1. Example of a worksheet: Distance from a point to a plane

After this phase (lasting about 15 minutes), all groups rotate their papers in a way that every group gets a term that is different from theirs. Now they need to read the other group's definition and make adjustments, if needed. A new, improved definition should be written on another blank piece of paper.

After another 15 minutes, groups make the last rotation, so every group is now introduced to all three terms. The final task is to make the manipulative model based solely on the definition that other students came up with. We provided different kinds of materials for students: cardboard for representing planes, styrofoam balls as points, different size skewers for lines, glue, rope, scissors, etc.

After another fun and creative 20 minutes, one student from each group comes in front of the class to present the model they made based on the given definition. Their definition is pinned with magnets on the board while the student presents their work.

Groups with the same term present one after another. In this phase, the teacher comments on their work and asks questions to see if their model is good for this definition or if they are missing some cases and conclusions. Everyone can participate in the discussion and give their own point of view, since they were all thinking about each term at some point.

After the discussion, the teacher hangs a formal definition on the board so everybody can see it and comments on whether those two definitions were accurate or had some deficiencies and need to be improved. Other four groups present their work in the same way.

Observation of the Study Lesson

After they got the worksheet, students started to investigate the problem by looking at the objects from their task. They were moving the chairs, opening and closing window panes and their textbooks, looking at the projector lens, etc. The teacher pointed out that they don't need to solve those specific tasks and measure the angles and distances, all they need to do is write a definition based on a given example. We noticed that some groups instantly started to solve the problems and measure the angle or distance, so later their definitions contained some specific terms and examples. In every case, those concrete situations were helpful for them and gave them the incentive to write down their own definition. Some students had interesting remarks and questions, such as is the line in three-dimensional space, or whether the walls are perpendicular to the floor.

The groups that had to write a definition for the distance from a point to a plane were done before the others, and the groups that were dealing with the angle between two planes had the most difficult task and therefore needed a little more time.

In the second phase, students had to read another group's definition and improve the parts that weren't clear to them. Some groups struggled with understanding what was written and experienced the importance of precise language in mathematics and what a difference a few words can make.

The most interesting and fun part of the lesson was making the models. The students were genuinely surprised by the amount of materials teachers provided for them and got the opportunity to be creative and work with their hands, since they don't do it often in high school. It was nice to see them having fun and working together successfully as a team.

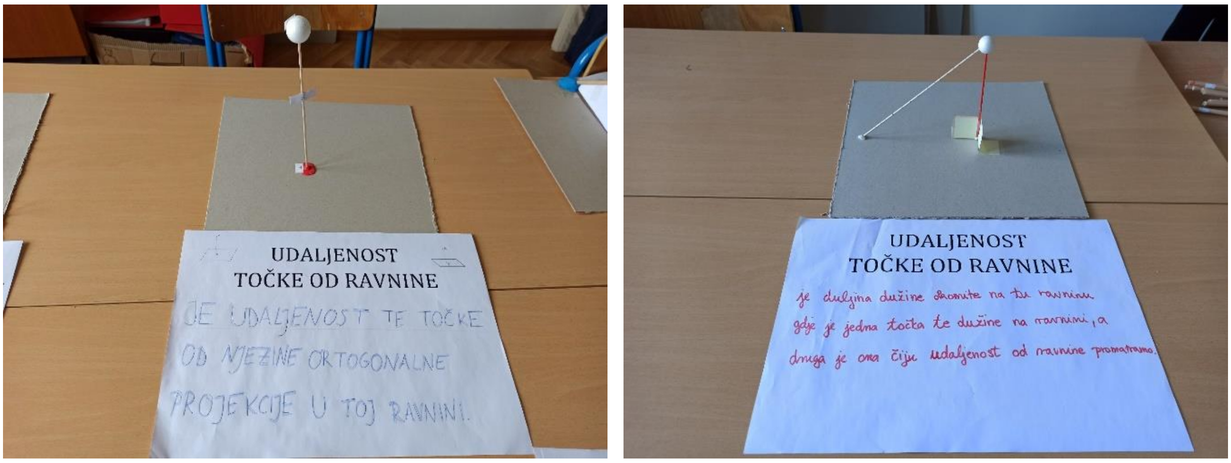

Figure 2. Examples of the students' work for "the distance from a point to a plane"

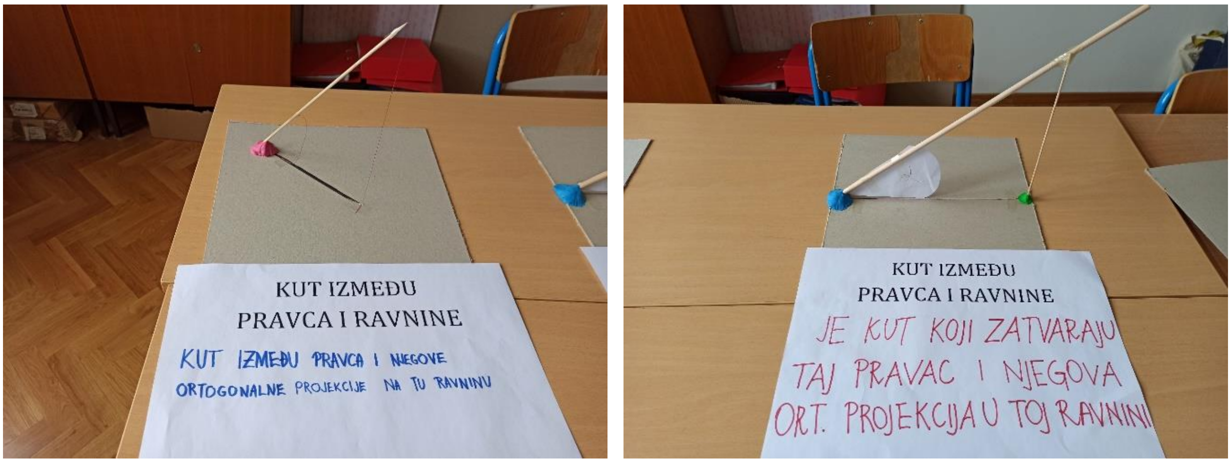

Figure 3. Examples of the students' work for "the angle between a line and a plane"

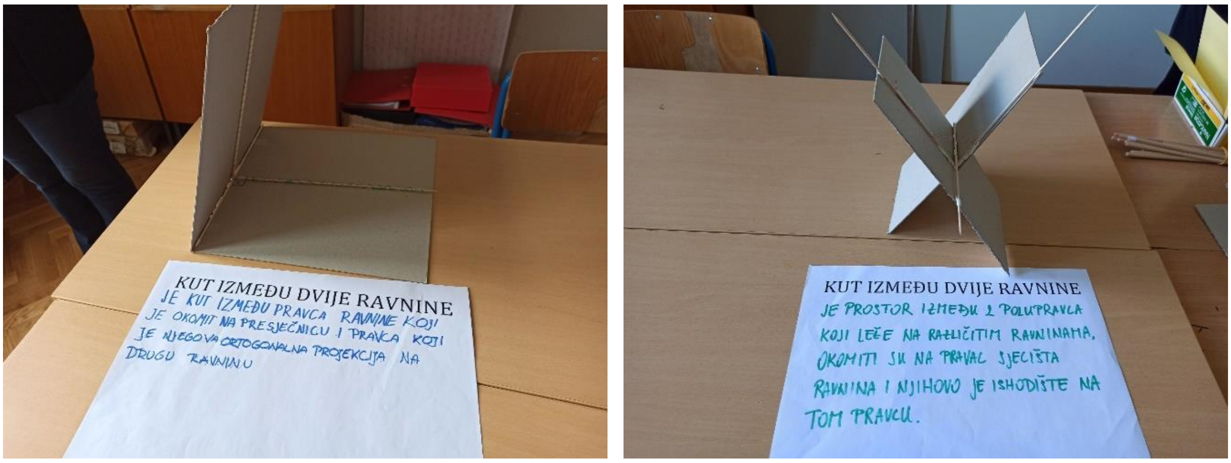

Figure 4. Examples of the students' work for "the angle between two planes"

After each of the three times we held this class, we took students' feedback. Here is some of it:

- *I learned a new lesson in an interesting way, so I understood it better.*
- *I was surprised by the simplicity of these definitions.*
- *It surprised me that as a group we came up with definitions that were quite accurate.*
- *I was surprised at how efficient we were and how logical and simple those definitions actually were.*
- *I was surprised at how fun it was.*
- *I learned how to function better in a team environment.*
- *I was surprised by the success of my group in given circumstances.*
- *It surprised me how challenging it was to understand the other groups' definitions.*

We were very pleased with their reflections and comments and concluded our goals were achieved.

Reflection

The lesson was very successful, most of the groups managed to define their terms correctly, although some definitions were not precise and were missing some observations and conclusions. But it was a great opportunity for a teacher to comment on their work, and accentuate the importance of precisely defining, and the beauty of simplicity, as our students said in their feedback.

It was interesting to see how all the groups really enjoyed making their models and were sincerely surprised when they realized most of their definitions were correct. All six groups experienced difficulties and challenges in understanding someone else's definition, since they were put in the role of a reviewer. They saw the need for using precise mathematics language which was very educational for them. And finally, they had a challenging task to make a model based on someone else's definition, and it was again something they don't usually do but has many benefits for their better understanding.

We have noticed that some groups needed more time than others because some terms were more difficult to define than others. For example, the angle between planes was the most difficult term for students, and the distance from a point to a plane was the easiest one. But since every group had the opportunity to work on all three definitions, the overall dynamics weren't disturbed, and they ended up having an equal amount of work.

We will definitely repeat this lesson study since it was so successful, and we look forward to seeing how this lesson would contribute to their future problem-solving tasks in which those definitions are needed. We surely hope it will improve their reasoning and give us teachers a good reference point in the lessons ahead.

Argumentation

Discovering algebraic identities in geometric figures

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Identifying the problem and learning goals

Do we know how to precisely, briefly, and clearly explain any of our claims or ways of reasoning? Do we even need that skill at all? And how can we develop that skill? These are the questions that came up when we were thinking about what is challenging (not to say difficult) for our students in math lessons.

We noticed that they are quite clumsy in explaining the procedures they used to solve a task. They believe that it is enough to say "*Well, that's obvious*" or "*It can be seen".*

Therefore, we decided that the research topic of our teamwork should be argumentation, i.e., we wanted to work with the students on developing the ability to clearly, systematically, and precisely express their way of thinking and reasoning.

Planning and creating the lesson plan

Guided by the frequent student response "*Well, that's obvious!*", we decided to confront our students with a task in which they must connect a simple geometric drawing with a mathematical formula. Formulas, which are in the background of the given drawings, are well known to students. That's why a lesson conceived in this way could be understood as a kind of repetition of the previously learned content.

In the first part of the lesson, when the students encounter something (at a first glance) new, the focus is on inquiry, connecting different contents, and mutual communication within the group.

In the second part of the lesson, the presentation of the results, the emphasis is on the reasoned explanation of the solution. We will consider the class a success if the communication between the students is high-quality and constructive and if during the presentation the students manage to explain their solution and the way they came to it to the class.

60 minutes are planned for the implementation in the class (if necessary, it can be 90 minutes). Materials (five worksheets - four for class work and one for homework) have been prepared and are presented in Figures 1 and 3.

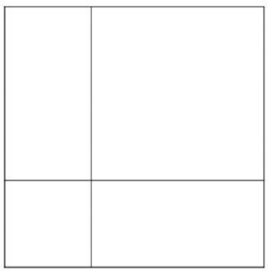
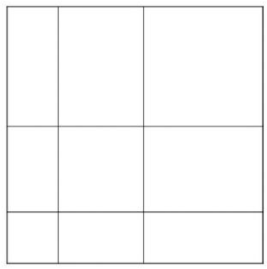
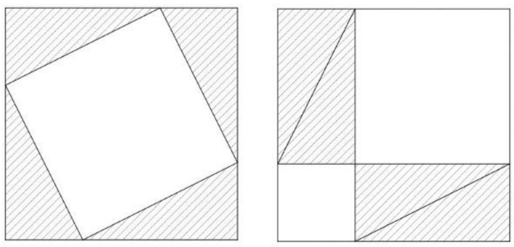
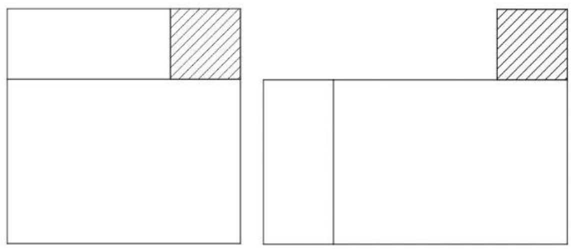
Worksheet 1	Worksheet 2
	
Worksheet 3	Worksheet 4
	

Figure 1. Worksheets for classwork

The class is divided into eight groups of three or four students each. Each group receives one of four sheets of A4 format paper (two groups work on the same task). The students are given a common instruction:

The task of each group is to recognize the mathematical story behind the obtained drawing, i.e., they should connect the obtained drawing with some mathematical fact.

Students explore independently (for 20-25 minutes). In this phase, the teacher walks around the class and observes the development of the situation. Some students will immediately recognize the formula in the background of their drawing, while others may look for a mathematical connection by measuring. If they stay at the level of concrete numbers for too long, the teacher asks the students after about fifteen minutes what conclusion they reached using the measured data. Then she suggests that they try to change the data in the picture and see if the same conclusion is valid then. Can they generalize their reasoning – perhaps using general labels instead of specific data?

Students are instructed to prepare for the presentation of their conclusions, while the teacher prepares the board. She attaches the worksheets to the board by magnets and allows each group enough space to explain the solution to their task.

Figure 2. The board before the presentation of solutions

After ten minutes, representatives of the groups that had the same task come to the board and write down their solutions in the designated place. In doing so, they comment and explain how they came to their conclusions and, if necessary, answer the questions of the other students in the class who are listening carefully. If there are several different ways of solving the task (from different teams), all the ways are shown. If a team's explanation is unclear or incomplete, the teacher uses sub-questions to help the team representative to show their joint work as clearly as possible. It may happen that some groups fail to reach any conclusions. Then the group representative tells what difficulties they encountered during their research. At the end of the presentation, all student results can be seen on the board.

Reaching the end of the class, the last five minutes are planned for writing feedback. We believe that this is very useful because the students are immediately aware of their experience during the lesson and able to write down their comments and thoughts about this type of teaching.

Then they get a new task, presented on the worksheet in Figure 3, the solution of which each group should hand in at the next lesson.

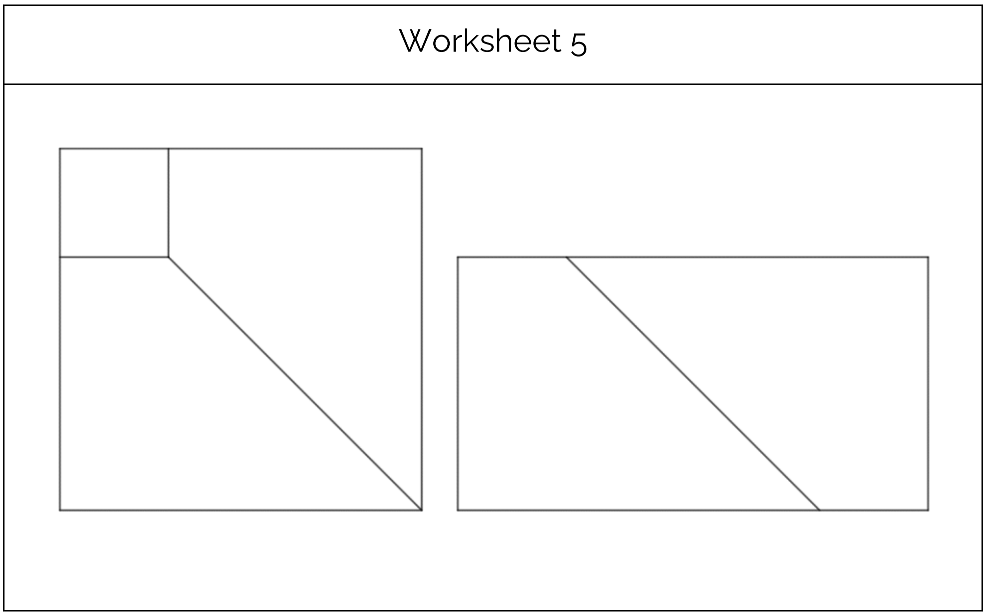


Figure 3. Homework task

Observation of the Study Lesson

After receiving the worksheets, the students began to investigate which mathematical fact could be hidden in the given drawing. They approached the task in different ways. At first, they just looked at the picture, turned it in various positions, and then they started to notice some lengths of the same length, they marked them with the same labels and gradually came to the areas of the rectangle. In other groups, students observed various symmetries, and supplemented the drawings with diagonals – they observed and discussed whether they could conclude something from it. Some groups started with a general notation for the lengths (e.g., *a, b, c*) and then looked for patterns that connect them. All groups lively discussed: What could it be? What is hidden there? As someone within the group noticed something, he tried to convince all the other members of the group. One could hear: "*Look, this area is equal to this one, so when we add them up we will get...*", or: "*That could be a Pythagorean theorem, because we have a right triangle, we need to mark the legs and the hypotenuse and then write it down...*".

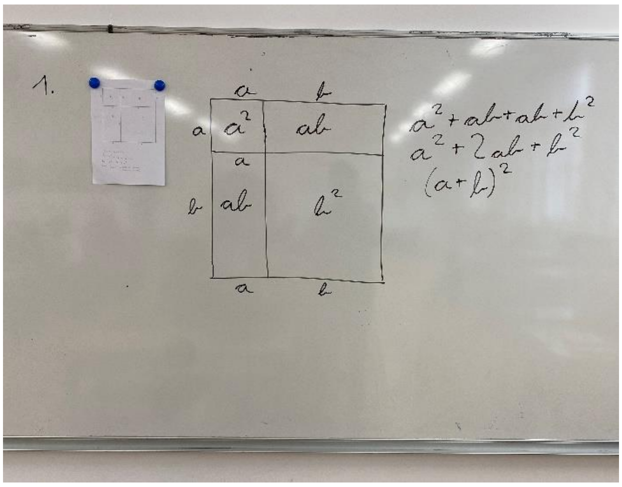
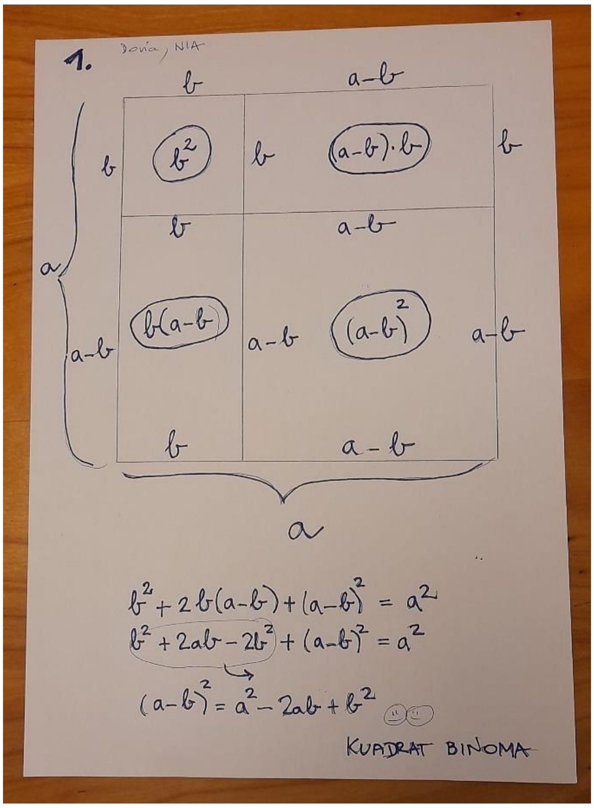
The teacher visited all the students, listened to their discussion, asked a few questions and watched them write down the solutions. As expected, some groups started to measure the sizes on the offered pictures, so the teacher acted according to the plan, i.e., after about fifteen minutes, she instructed those groups to try to generalize the measurement and calculation with concrete numbers. The discussion continued. From the initial "disorientation" came the introduction of the symbols *a, b,* and *c* for the lengths of the sides and the recognition of familiar expressions.

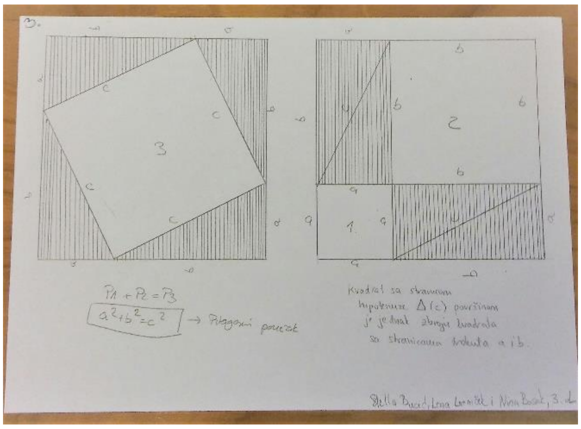
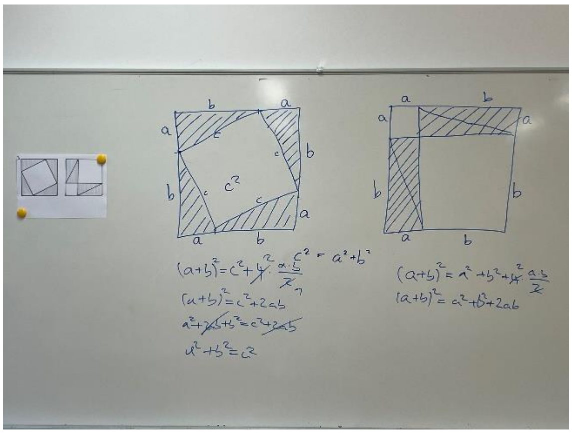
Then the students were instructed that the representative of the group should prepare a presentation of their solution or their work. Before the student presentations, the teacher prepared the board. She divided the board into four areas and thus provided enough space for each group to explain their solution to the task.

During the presentation, the students tried to systematically explain their solutions and write them down on the board (at the same time, the first graders pleasantly surprised us with their precision of communication). Everyone listened carefully and asked questions if they did not understand something. If the group did not find a solution, its representative presented the reasoning process and described the difficulties they encountered in doing so.

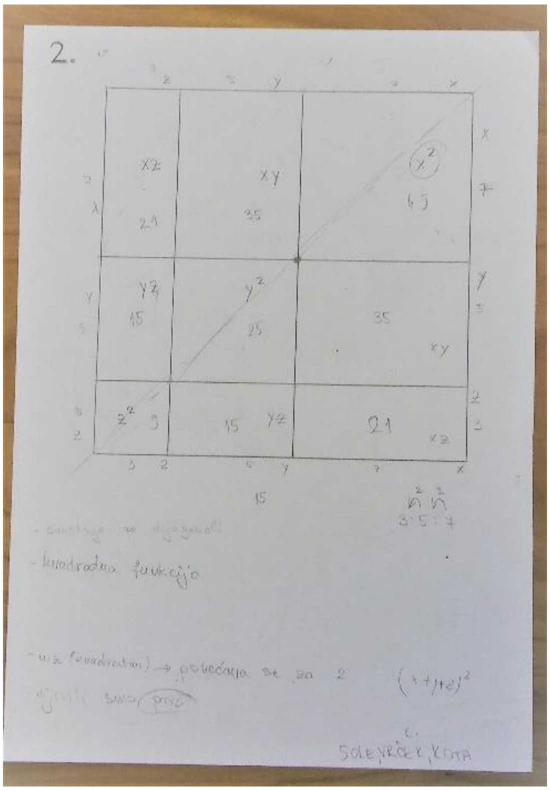
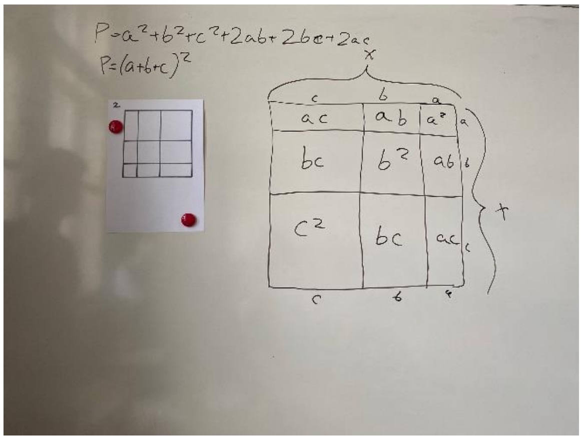
The teacher was the moderator of the process - she asked sub-questions when necessary and thus helped the students to make the argumentation of the solution clear, precise, and correctly written.

In all the classes in which we held this lesson, the groups that had worksheets 1 and 3 successfully solved their task (in just 20 minutes).

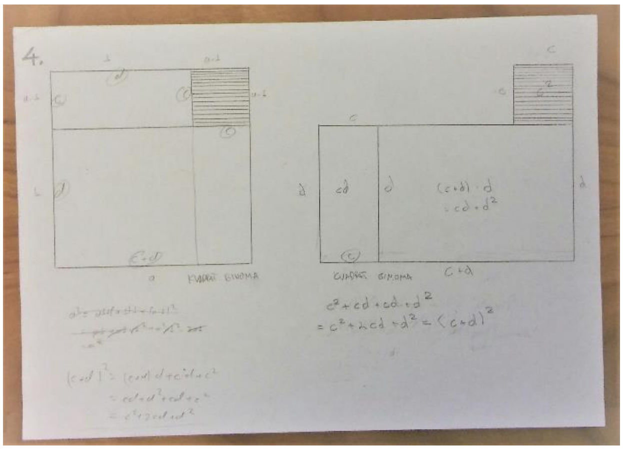
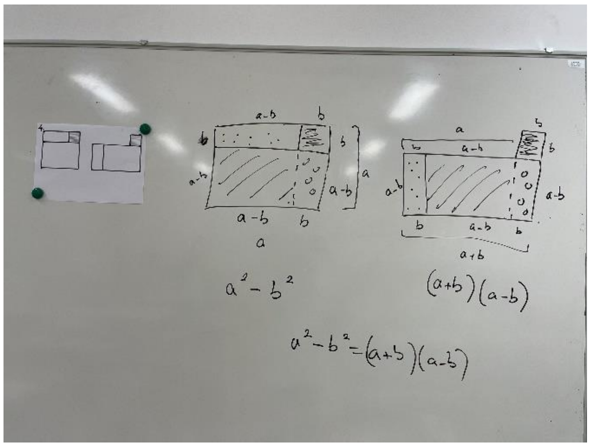
Worksheet 1 (the square of the binomial sum)	Worksheet 1 (the square of the binomial difference)
	
Board after the solution presentation	We didn't expect this, this result surprised us
Labels a and b were introduced so that the initial figure is a square with a side of length $a + b$. It is divided into two smaller squares and two congruent rectangles. Equalizing the areas, it follows: $(a + b)^2 = a^2 + 2ab + b^2$	Labels a and b were introduced so that the initial figure is a square with a side of length a. It is divided into two smaller squares – one with a side of length b, and one with a side of length $a - b$, and into two congruent rectangles. Equalizing the areas, it follows: $(a - b)^2 = a^2 - 2ab + b^2$

Worksheet 3 (Pythagorean theorem)	Worksheet 3 (Pythagorean theorem)
	
Standard labels were introduced for the lengths of the legs of a right triangle, a and b, and c for the length of the hypotenuse. First, it was observed that the shaded right triangles are congruent, which means that the "white" areas in both squares are equal, i.e., that $c^2 = a^2 + b^2$ holds. Note: The "white" figure in the left square is certainly a rhombus, and it must be proved that it is also a square (with a side of length c)	After introducing the labels, a, b, and c, the students presented the area of both squares as the sum of the areas of their parts. Equalizing the areas of the left and right squares, the Pythagorean theorem is obtained.

The groups with worksheet 2 needed more time. There, the students completed the pictures, drew the diagonals of the rectangles, and noticed the symmetries - and then in a mutual discussion or supported by the teacher's instructions, they discovered the solution to the task. The decisive moment for the recognition of the formula was when they introduced labels for the lengths of the sides and used them to write down the area of the observed rectangles.

Worksheet 2 (the square of the trinomial sum)	Worksheet 2 (the square of the trinomial sum)
	
In this example, we can see drawn diagonals and an attempt to observe symmetries. Then the pages were measured and the concrete areas of the figures were calculated. After that, labels x, y and z were introduced so that the initial figure is a square with the side of length $x + y + z$. It is divided into three smaller squares and six rectangles. Equalizing the areas, it follows: $(x + y + z)^2 =$ $= x^2 + y^2 + z^2 + 2xy + 2xz + 2yz$	Labels a, b and c were introduced so that the initial figure is a square with the side of length $a + b + c$. It is divided into three smaller squares and six rectangles. Equalizing the areas, it follows: $(a + b + c)^2 =$ $= a^2 + b^2 + c^2 + 2ab + 2ac + 2bc$

When assigning the tasks, we were very careful to which group we would assign the worksheet 4. In our estimation, it contains the most challenging of the four problems, as it offers several possible interpretations. This was confirmed in the lessons held in all classes. Two classes managed to discover the expected solution. To our surprise, one first grader found a different interpretation of the problem. And in one class, the students failed to discover a single "mathematical story" connected to that worksheet.

Worksheet 4 (the square of the binomial sum)	Worksheet 4 (the square of the trinomial difference)
	
Labels c and d were introduced so that the left figure is a square with a side of a length $c+d$. Equalizing the areas, it follows: $(c+d)^2 = c^2 + 2cd + d^2$	The left square has a side of length a, and the smaller square inside it of length b. We observe the areas of the figures obtained by removing the smaller square. Equalizing the areas, it follows: $a^2 - b^2 = (a-b)(a+b)$

After the presentation of the solutions, the students were given a task - worksheet 5 (Figure 3), the solution of which they should submit to the teacher in the next lesson.

The goal of the lesson - for students to try to articulate their thoughts and to convey them to others - has been achieved. Everyone liked this form of work. You can see for yourself if you read the students' reviews:

... "I think that the group work was very instructive because it encouraged us to think and cooperate together. It's also great that we commented and explained the work of each group immediately after, while everything was still in our heads."

... "For me personally, it's very fascinating to see the way of thinking of other people in our smart class and then apply all our thinking to reach a common goal."

... "The lesson passed quickly and it was fun. I remembered everything and it was a great feeling to come to a solution alone, i.e., in a team. I think it's a great way of learning and I'd definitely like to do it more often."

... "I think this was very useful and fun. We moved away from normal classes and faced problem-solving. We worked in groups alone, but we could always ask you for help. We had a certain amount of independence, but again we didn't feel left to our own devices. I would like to do this more often because I think this is a good way to encourage argumentative and logical thinking."

Reflection and concluding remarks

We held the same lesson several times and each time the objectives of the class were achieved. The students researched together, and it was nice to see the excitement of solving the problems even in those students who are normally not very active in class. Most groups were able to find the mathematical formula associated with the given drawing; for some of them it took a little longer to find a solution, and some (really, the minority) failed to solve the problem even though they tried. In the end, they presented their discoveries in front of the class and neatly wrote them on the board. It should be emphasized that sufficient time should be provided for the presentation of solutions, discussion, and possible comments. That part of the lesson is important, so it should not be rushed. Moreover, if necessary, the time allotted for argumentation can be extended.

The teachers (the lecturer and the observers) carefully followed what was happening in class and paid special attention to the questions they asked themselves before the lesson: Will the students manage? Will they be able to connect the unknown images with the known formulas? How will the students approach the tasks? How will they communicate? What instruction will the teacher give to the students without affecting their research? How will the teacher help the students if he notices that the argumentation is unclear or incomplete? Will we be satisfied with the observed lesson?... And yes, we are very satisfied with this lesson! And we have already answered all the questions in this report.

This experience highlighted to us how useful it is to change teaching approaches because in that way each student can find the way of learning that is most acceptable to him/her. And we teachers have learned that not everything should be presented to students, instead we can discreetly guide them in their independent discovery of solutions.

The students were also satisfied with the lessons. They pointed out that it is useful and fun to research in a group. They also mentioned that they like to hear how their colleagues think - it inspires them and encourages new communication and learning. For most, the most important moment was the discovery of the formula associated with the drawing - they are satisfied and proud to have discovered it independently. Several students also mentioned the importance of good argumentation - and that is exactly what we wanted to achieve.

How does it work?

Understanding the inner working of mathematical induction

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Identifying the problem and learning goals

Students often perceive the principle of mathematical induction as a procedure. At the same time, they do not understand why it is necessary to carry out all the steps in proving by mathematical induction, what are the role of induction basis and the induction step. For many students, it is difficult to build the induction step based on a statement that has not been proven, namely the induction hypothesis. They also do not recognize mathematical situations in which this principle is useful and applicable.

This misunderstanding manifests in a way that students "forget" to check the induction basis or check the basis for $n = 1$, although number 1 is not the smallest natural number for which the statement holds, or assume that the statement holds for every natural number, and so on. Our goal was to find, design, and implement an approach that would reduce these misunderstandings.

Planning and creating the lesson plan

Two periods (90 minutes) are planned for the implementation of the Mathematical Induction teaching unit. Students work individually, in accordance with epidemiological measures. Worksheets and a textbook are used as teaching materials. At the beginning of the class, the students, following the example of several steps already taken, check the given statement for consecutive natural numbers and answer some questions. The students are expected to connect the two steps, i.e., to notice that if the statement is valid for some natural number, this entails that the statement is also valid for its consecutive number.

$n=1$	Check: $1=\frac{1\cdot(1+1)}{2}$
$n=2$	$1+2=\frac{1\cdot(1+1)}{2}+2=\frac{2+2\cdot2}{2}=\frac{2\cdot(2+1)}{2}$
$n=3$	$1+2+3=\frac{2\cdot(2+1)}{2}+3=\frac{2\cdot3+3\cdot2}{2}=\frac{3\cdot(3+1)}{2}$
$n=4$	
$n=5$	

Figure 1. The table in which students check a familiar statement

The students record their answers on sticky notes, and after reading them, the teacher selects some and posts them on the board without comment. The next task for students is to read the principle of mathematical induction in the textbook, study the solved example and prove one statement on their own using this principle.

After that, students get questions about understanding the principle of mathematical induction. The teacher leads a discussion in which she tries to connect these answers with the answers highlighted on sticky notes and to make students aware of the meaning of the induction basis, hypothesis, and induction step.

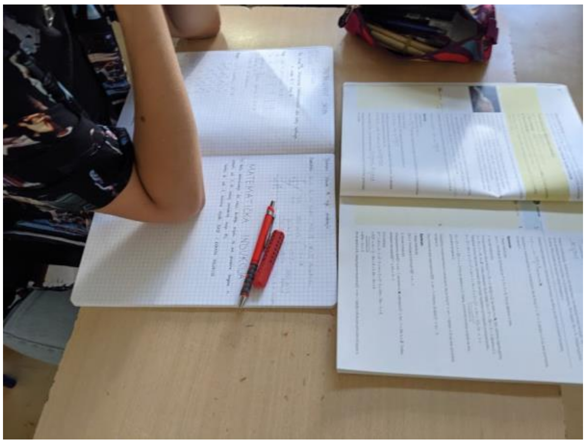

Figure 2. Reading the textbook

Finally, on some solved examples in which the students need to check whether the proof was carried out correctly, it will be determined whether the students understood the important parts of proving by mathematical induction.

Observation of the Study Lesson

Figure 3. Students working individually

In the beginning, the students were very active in filling in the table and carrying out the required steps. According to their responses, most seemed to intuitively connect two consecutive steps:

"For step n=5 we have to use the results for n=4."

"They are connected by the same formula."

"We can prove each line with the previous one."

"Yes, because we see a pattern."

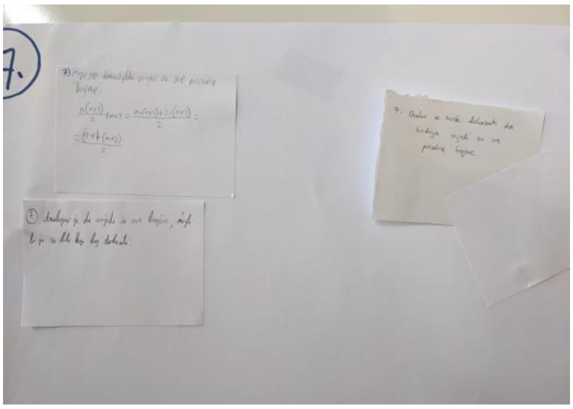

Figure 4. Some students' responses

However, most students found it difficult to formulate this connection using mathematical language, as well as answering the question: "*Can it be proved in this way that the statement holds for every natural number?*"

Some have only written "*Yes*". This was an interesting answer: "*In theory it is possible, but there are infinitely many numbers*." We even got the answer: "*If we assume that it holds for n and prove that it holds for $n + 1$, and we calculate that it holds for $n = 1$, then it holds for any n from $\mathbb{N}$.*"

During the discussion, the teacher reminded the students of the difference between deductive and inductive reasoning in logic. All students noticed that we use the principle for proving a statement for every natural number, but some said that it is a theorem and some a procedure. Students had difficulties in formulating answers about the role of the induction basis and the induction step. One of the frequent answers was that "*the induction step serves to connect two consecutive numbers, to show that if something is valid for the predecessor, then it is valid for the successor*", and one student mentioned that "*the step is important for achieving a domino effect*". For the induction basis, one of the answers was: "*because the induction step results from it, and this leads to the fact that $T(n)$ is valid for all n*".

Reflection and concluding remarks

The students filled in the table without any difficulties. It was clear to them that in the proof for some natural number (from $n = 1$ to $n = 5$) they should use the statement they had checked for the predecessor of that number. Also, for greater n, they realized that they did not have to check the validity of the statement for every natural number lesser than that number, but that they could assume that if they checked it for the predecessor, then they would use it to prove for the given number.

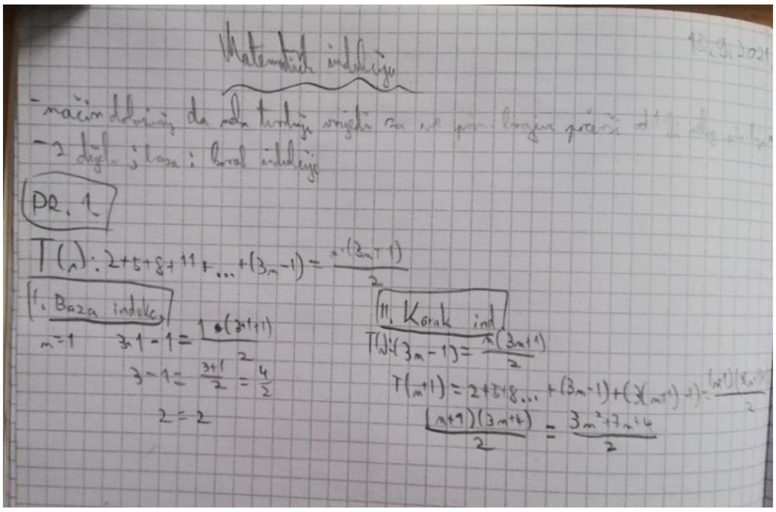

Figure 5. Students' work

Students intuitively understood the induction step, but it was difficult for them to formulate and state, i.e., write the whole proof by mathematical induction. Using a lot of sub-questions in the discussion, the teacher insisted on the concept of induction steps and the necessity of the basis, i.e., that the students formulate on their own the principle of induction and understand the importance of all steps in the proof. Most of the students were able to understand the necessity of the basis, hypothesis, and step and state the principle of mathematical induction.

Examples in which students needed to check whether the proof was carried out correctly contributed to some students realizing which mistake they had made in the task they were solving on their own. In this task, they often omitted the hypothesis of induction or in the hypothesis they wrote that the statement is valid for all n.

It seems that reading a mathematical text with its content unknown to students contributed to the conceptual understanding of mathematical induction. The task was mathematical, with a lot of content, but that was not an issue as it was meaningful to students.

The radian-meter

Defining trigonometric values

Aneta Copić, Darja Dugi Jagušt, Marina Ninković, Vesna Smadilo Škornjak, Eva Špalj

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Identifying the problem and learning goals

For teachers in our school, learning with understanding for all students is very important. We also put emphasis on mathematical communication: formulating statements and conclusions and proving.

After the students have adopted the concept of trigonometric values of real numbers and after the trigonometric functions were defined, the question arose: how can we now take an angle instead of a number for the sine argument?

We use calculators to calculate the trigonometric values of angles in polygons (triangles, quadrilaterals, etc.). Students quickly accept it, but some of them get confused when working with a calculator because they are not sure when to calculate in radians and when in degrees. We noticed that there is a problem with the transition from the trigonometric value of a number to the trigonometric value of an angle.

The aim of this lesson is to expand the definition of the sine (cosine) of a number to the definition of the sine (cosine) of an angle, without using a calculator, using only a radian-meter (a trigonometric circle with marked numbers on the circle).

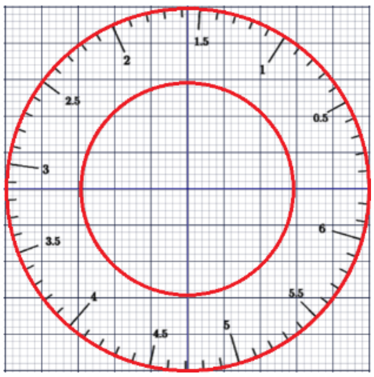

Figure 1.

The radian-meter

The target knowledge for this Study lesson, planned and tried in a third-grade class, is to expand the definition of a trigonometric value of a number to a trigonometric value of an angle, with broader objectives: mathematical communication - answering to a question: What is a Sine of an angle?

Planning and creating the lesson plan

The lesson and the materials are prepared for both in-person and online learning environments. Since the lesson is an introduction to the trigonometry of a triangle - sine and cosine rule, a time of 40 minutes is allotted.

At the very beginning of the lesson during the distribution of materials and giving instructions, the teacher is discussing the measurement of the angle in radians as a necessary introduction to the inquiry part of the lesson.

Two activities are planned for students in which they will work individually. In doing so, they will use a transparent foil with a *radian-meter* as a measuring tool.

For the first activity, students will get three different polygons, cut out from the paper:
- right-angled triangle with a hypotenuse with the length of 1 unit
- isosceles triangle with legs having the length of 1 unit
- concave quadrilateral whose two sides that form a reflex angle have a length of 1 unit.

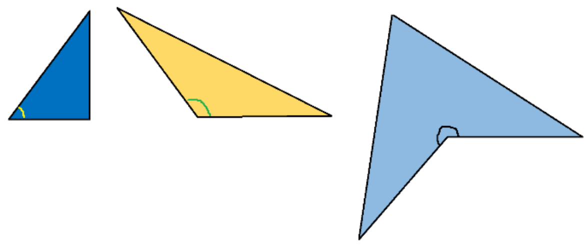

Figure 2. Different paper shapes with one marked angle

Students will use only the obtained *radian-meter* to determine the trigonometric values of the marked angles in the obtained shapes. Thus, students must place the obtained shape in a position that will allow them to determine the sine and cosine of marked angles, using their knowledge of the unit circle.

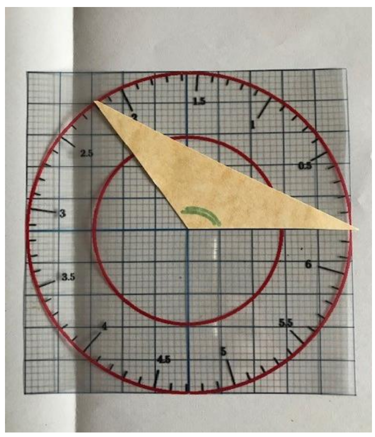

Figure 3. Placing a triangle on the radian-meter

The teacher has prepared a presentation in *The Geometer's Sketchpad* program to help with the discussion and institutionalization, after the first activity.

For the second activity, they get an obtuse triangle with one side having a length of 1 unit. Another side is marked and its length must be determined by the marked (obtuse) angle (Figure 5). This activity is planned with the intention that students apply the results of the first activity, and as a motivation for the next lesson.

To evaluate whether the goal of the lesson has been achieved, at the end of the lesson students had to submit the answer to the question: *What is a Sine of an angle*?

Observation of the Study Lesson

Task 1 was to determine the trigonometric values of the marked angles.

Almost all students did what was expected and measured trigonometric values of the marked angles as abscissa or ordinate of the point on the circle.

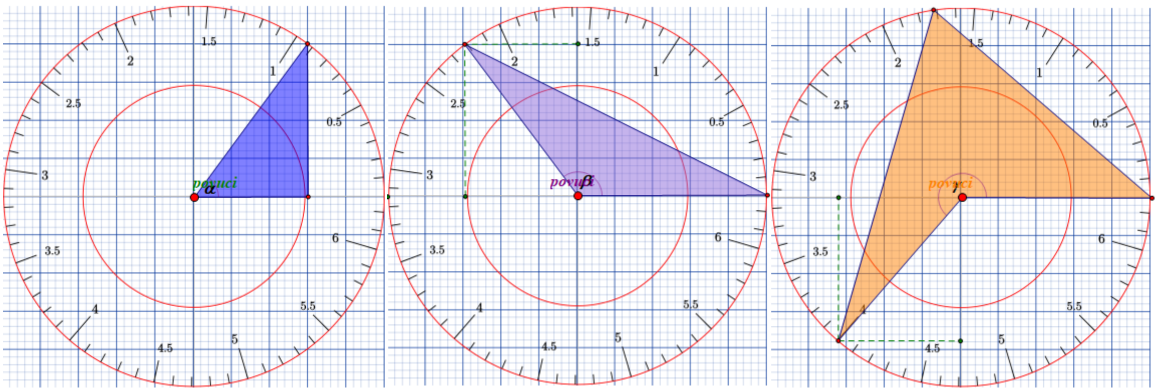

Figure 4. The position of polygons on the radian-meter – online lesson

The students didn't manage to argue mathematically why such a procedure is justified for an obtuse triangle and the teacher's intervention was necessary at that moment. After the discussion the teacher concluded:

> *We can equate the measure of an angle with the length of the circular arc of a unit circle. The sine of an angle whose measure in radians is α is equal to the value of the sine function calculated for the number α.*

Task 2 was to express the length of the marked side in a triangle by trigonometric values of the marked angle. Unfortunately, there was not enough time to discuss all the reasonings of the students.

The students were asked to write the definition of the sine of an angle in their own words. This activity was performed at the very end of the lesson and it would take another lesson to discuss their answers. Some of them are:

It is the sine value of the real number; that number is equal to the angle in radians.

The ratio of the opposite leg and the hypotenuse.

The sine value of an angle (measured in radians) is equal to the value of the function for that angle.

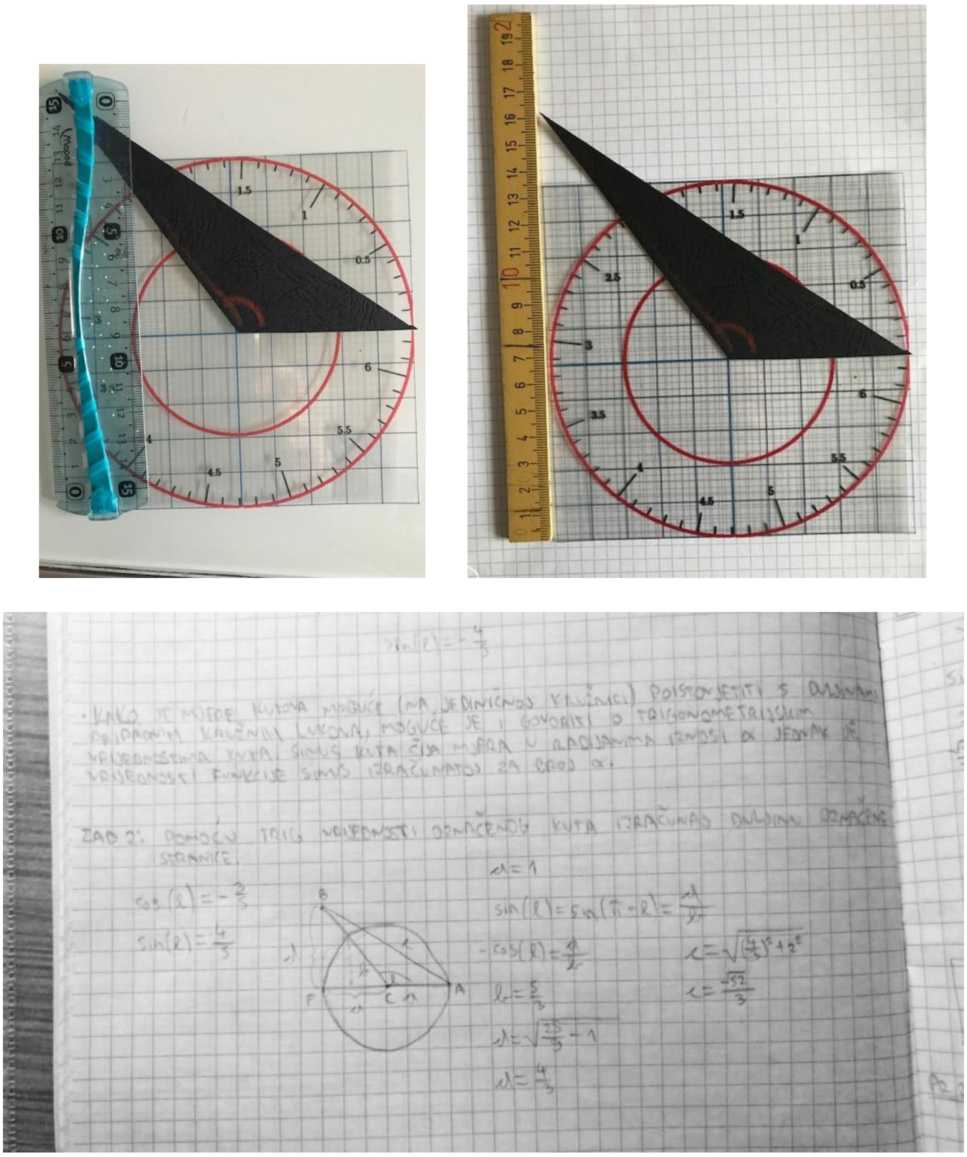

Figure 5. Some student attempts

Reflection

The selected teacher explained that some decisions to change the lesson plan were made based on the time constraint. In particular, the students were a bit confused by the first task, making them hesitant to give a concise answer to the question posed, which is why the teacher had to encourage them to speak. This was later identified as too much guidance on her part. Also, the teacher had an impression that, since this class was familiar with the radian measure of the angle and the unit circle definition of the trigonometric functions were mentioned earlier, better students maybe had a problem understanding what exactly the question was here.

One suggestion was that instead of going to the second task, more time could be devoted to discussing the definition and improving students' mathematical communication.

In the second activity, teachers failed to predict that the students will have a problem with identifying the x-coordinate of point C as −1. Without that information, the task became more difficult and none of the students was able to finish it in the given time. The conclusion reached during the reflection is that the teachers all felt the second activity might be an interesting follow-through, maybe even with a lot more time on students' hands.

There was a discussion concerning the order of trigonometric topics in the previous curriculum (still conducted with this class) versus the order in the new, upcoming curriculum: in the previous one trigonometry of the right-angled triangle was taught in the second grade, while in the third-grade students switched to the trigonometric functions and then back to laws of sines and cosines (using DMS measure of the angle). In the new curriculum, both the right-angled trigonometry and the laws of sines and cosines are taught before the trigonometric functions are introduced, which might improve students' understanding of the difference between the two.

As the most important conclusion of the observed lesson, the teachers pointed out the importance of developing and nourishing the students' mathematical communication fluency, often using tasks and activities like the ones used in this lesson.

The sine of a sawtooth?

Connecting the circle definition and the graph of the sine function

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Identifying the problem and target knowledge

Students encounter the sine in the Dutch curriculum in at least four contexts: first in right-angled triangles, then in the context of more general angles (using a button on a calculator), then in the circle (the geometric definition), and finally as a height-angle graph (sinusoids). For students, it is challenging to relate these perspectives and to move from one to the other. Our brilliant explanations and flashy demonstrations in cool apps (GeoGebra) are unfortunately not enough to do anything about this. That is why in our lesson study we opted for an approach with old-school means in which the learner has everything in their hand - literally - developing the circle and graph perspectives through inquiry and relating these to the triangle perspective.

Planning and creating the lesson plan

The designed lesson is built around four tasks. A pivotal role is played by a disc with a dot (in our case made from an old CD), of which each group of pupils gets one. The teacher introduces the motion of the dot when the CD is rotated and elicits the question of how the height of the dot varies during that circular motion. The first task for the students is:

Figure 1. Old-school do-it-yourself CD with a dot sticker

Make a graph of the height of the dot against the rotational angle, on a sheet of paper.

The teacher does not give any further instructions. In the classroom hangs a clothesline on which the students hang their worksheets when they are finished. The intention is to make the various choices and (mis)conceptions visible. That there is some variation is evident when the students look at each other's work. They argue with each other: should it be some kind of a sawtooth, parabolas glued together, semicircles (some students draw the outline along the disk) or something else? Do you start with the dot at the top or somewhere else? Does the graph stay above the axis or pass through it? Should we use degrees or radians?

The teacher at first ignores the apparent misconceptions – leaving this discussion to the students themselves - and addresses a few conventions: the middle of the disc is level zero, rotation is performed counter-clockwise, and the radius of the circle is 1. Task 2 for the students is then the same as task 1. With more structure and comparison material, the students make a second attempt and hang the result on the clothesline next to their first work. After this, the teacher and students have a classroom discussion about the choice of an approach and of a graph which is better or best and why.

Observation of the Study Lesson

In the second round, we witnessed some groups stubbornly sticking to certain graphs: semicircles, parabolas, or sawtooths. It is understandable that students resort to forms they already know, and yet: The sawtooth and semicircles are less logical, if you observe more carefully.

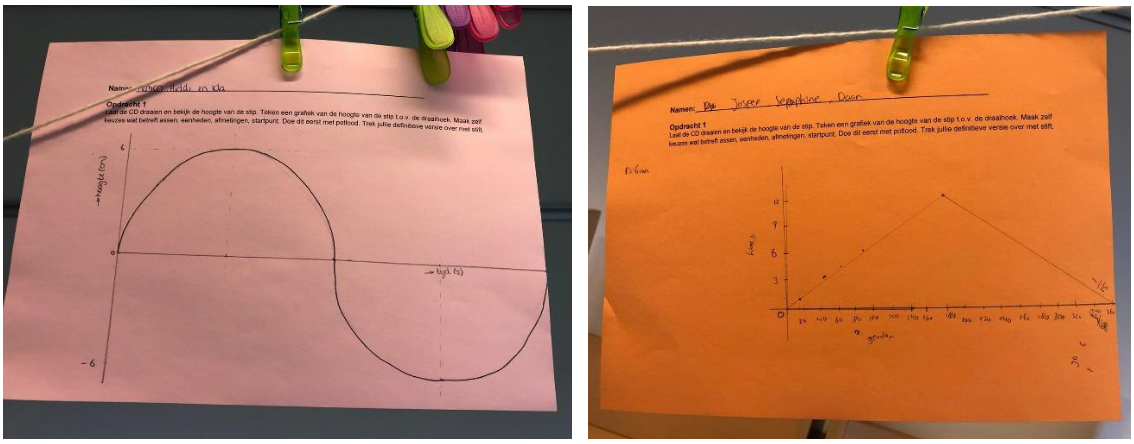
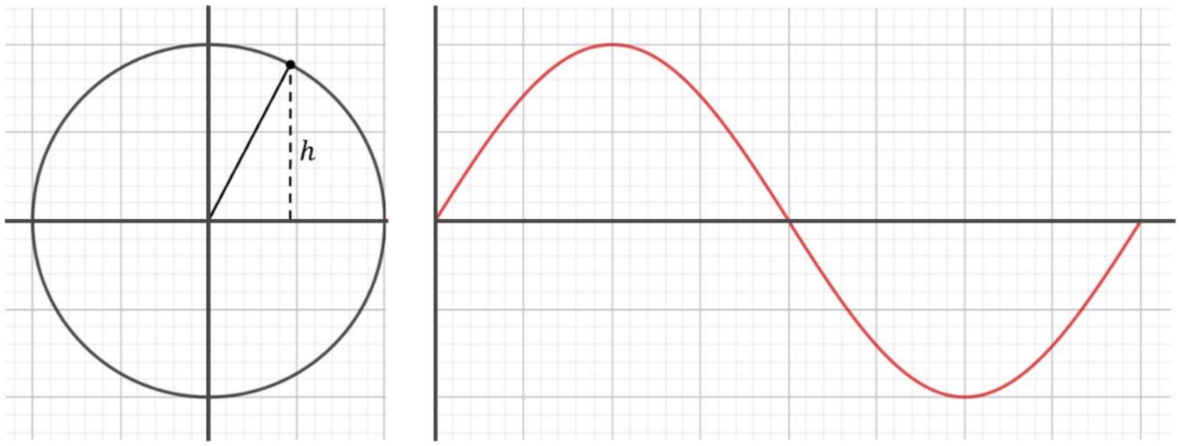


Figure 2. Students' graphs hanging on the clothesline

Fortunately, some students do try and introduce the (vertical) velocity of the dot into the discussion. Ultimately, the teacher guides the discussion to the correct solution, which is then shown, see Figure 3.



Above you see the unit circle and the graph of the sine function side by side.

a) Place the dot that you see on the unit circle also in the graph. Put the corresponding angle α in the unit circle and on the horizontal axis of the graph.

b) Express the height h in terms of the angle α. Find help with this in the unit circle.

c) Find $\sin 90°$ and explain why. Calculate exactly $\sin 135°$ and $\sin 210°$. Draw that in the unit circle and in the graph.

Figure 3. Worksheet with the circle and the sine graph

We learned a lot from seeing how much difficulty students have with tasks b) and c). Students have to put the α and 1 in the circle themselves, but after that, it is a standard case of the sine in a rectangular triangle. Of course, this is not routine work. Here it comes to the heart of the whole lesson, which is to integrate the different perspectives to the sine: triangles, circle, and graph. Our idea and hope is that the experience of their own inquiry, a joint discussion, and the confirmation by the teacher will lead to a stronger anchoring of that cohesion - even though in each class only a handful of students can complete the tasks independently.

The final task brings both a new element and a look back at the previous tasks: Investigate how the cosine is present in the unit circle and draw the graph – possibly using the disc.

Reflection

What we just described was the shape the lesson took in the end, but now let's describe the process towards it. The lesson has been taught three times, and each time observed by two or three of us. Afterwards, we took at least half an hour to discuss the observations. In lesson study, those observations are not about whether the teacher did well, but whether the design choices for the lesson worked out well for the students. Based on this, the design is adjusted. Small and large matters were discussed, too many to list all. For example, the use of radians in the exercise was difficult for the students, and that distracted them too much from the main thing - so next time only degrees were used.

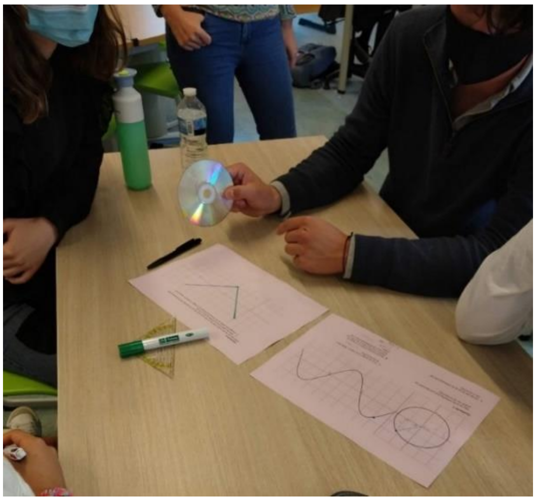


Figure 4. Pupils working with a disk and worksheet

We saw that students talked seriously about each other's graphs when they walked along the clothesline, but after that, too little time had been given for students to evaluate the differences and to determine together which graph could be the right one and why. This was then adjusted in the lesson plan. The formulations of the assignments were also adjusted a number of times, and the size of the groups went from four to three. It makes every lesson extra interesting to see if the adjustments are really improvements. We believe we designed a lesson that addressed the various perspectives of the sine in an integrative way. Through inquiry-type tasks students, as a group, successfully developed these insights. Now let's hope they endure!

Unfolding sin(2x)

Trigonometric angle-doubling formulas by folding papers

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Identifying the problem and target knowledge

The angle-doubling formulas in trigonometry are often an instrumental tool, but can also be a nice application of mathematical reasoning. For students, however, the formulas usually remain difficult, that suddenly pop up or are needed during solving an integral, equation, or angle calculation. Can you nevertheless get students to discover the formula for sin(2x) so that they remember it better and, moreover, experience something of mathematical inquiry? We made an attempt, inspired by origami, as a basis for mathematical explorations.

Planning and creating the lesson plan

The lesson plan was initially designed for a lesson of approximately 60 minutes in grade 11 (Dutch 5vwo mathematics B) and then optimized during three lessons at USG. The following report uses experience from the three lessons, but follows the chronology of the last lesson. For example, one of the adjustments was the discussion of the question: is it true that for each function f it holds $f(2x) = 2f(x)$ for all x? Originally, this came up halfway through the lesson, but now it has been used as the starting question for the lesson.

At the beginning of the lesson, we hand out coloured paper rectangular triangles (not isosceles), a different triangle for each person within a group. Half of the students were at home due to corona. There was enough room in the room for the students to work in groups of three and, in addition to the teacher, two other teachers and a representative from Utrecht University were present for observation. Before students started working with the task sheet, the assignment was introduced.

Observation of the Study Lesson

First, the class was asked what the equality $f(2x) = 2 \cdot f(x)$ means and whether it is always true. After trying out an earlier lesson plan, we have chosen to provide context for the problem with this question right at the beginning of class. Students may name some functions, such as linear, quadratic, and logarithmic, and check to see if it is true. What about sin(x)? Is it true that sin(2x) = 2sin(x)? Students are given a few minutes to think about this in their groups. Several groups quickly gave a counterexample (e.g., $x = 90°$). One student said: "*Instinctively, I say no*". A groupmate: "*Probably the answer is yes, otherwise they wouldn't ask*". A group suspects sin(2x) = 2sin(x). The plenary conclusion is, "*No, that's not always true*". Then follows the central question for the lesson: but how do you express sin(2x) in terms of sin(x)?

The definitions of trig ratios (SOSCATOA) were briefly reviewed in a triangle with angle α, hypotenuse 1 and rectangular sides cos(α) and sin(α). Then the worksheets were handed out and without further introduction, students were asked to do the first tasks.

With the instruction to call the acute angle α, students started looking for what they know about the other (folding) angles and whether they can also find 2α in the triangle with all the folding lines.

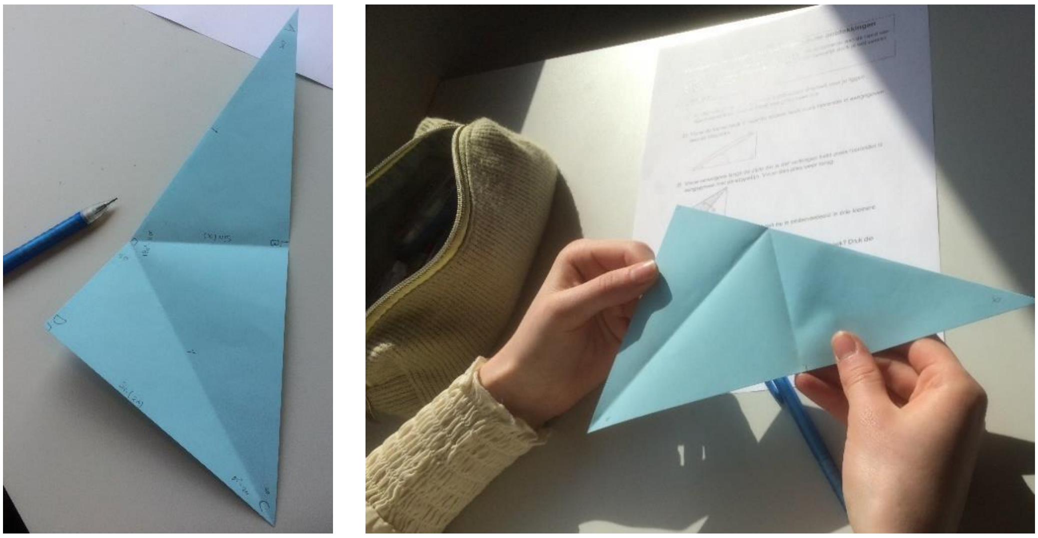

Figure 1. Following the folding instructions for a triangle

Folding instructions don't always turn out to be trivial, but together with each other, they came out okay. Each new angle found felt like a victory. Quickly, by folding, students saw another angle that is also α degrees and found an angle which is 180 - 2α degrees. Occasionally, some students think they can't go any further. The teacher walked around and clearly showed them that they can do the puzzles themselves. In the end, almost every group managed to find 2α. Is folding essential here? We are convinced that folding helps students to find right angles, equal angles and equal sides.

By now about 20 minutes have passed and the teacher took the floor. She drew the triangle with the folding lines on the board and asked what we now know about the triangle. This came at exactly the right time. Everyone was sufficiently up to speed and could follow the reasoning. It appeared that different groups followed different strategies and were happy to share them with the rest of the class. For a moment they took ownership of their research.

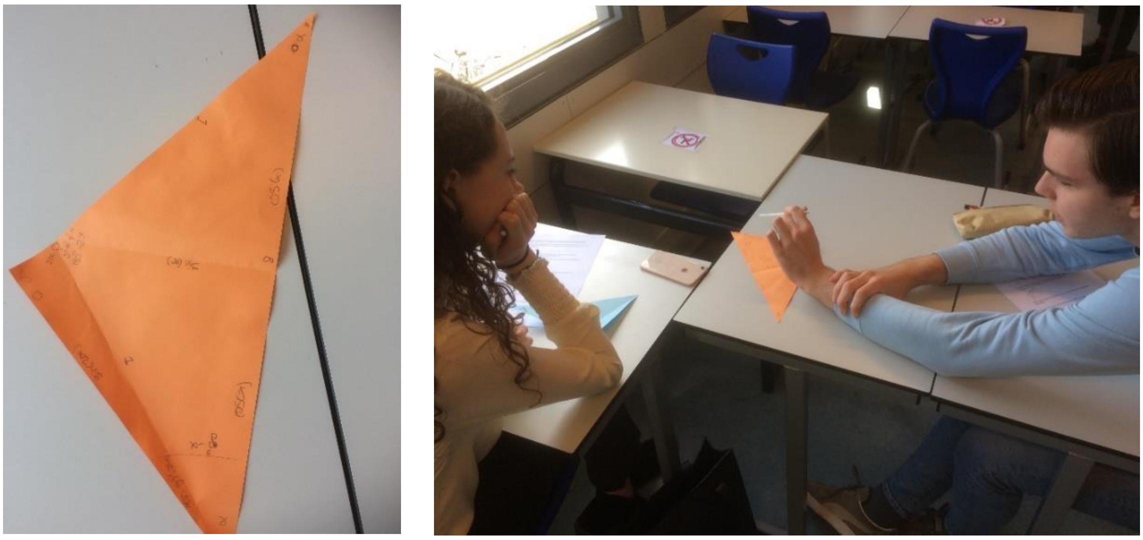

Figure 2: Identifying angles and similarities

We returned to the central question: What do we know about sin(2α)? This was the final task for which students were given 20 minutes. A few groups thought they could get through by noting that sin(2α) is the length of one leg of the triangle. The teacher replied:

> *"No, I am not satisfied that quickly. A formula using α and sin and cos. That's the challenge for you."*

Students, after discussing among themselves, dared to give line segments the length sin(α). In their search, they were getting closer and closer. Now it was again good that all individuals had their own triangles to prevent them from looking at concrete numbers/situations too quickly. The groups were all puzzling again. One group was in danger of dropping out. In that group was a noisy student who would rather draw Donald Duck than a triangle. With some encouragement, they got back to work. The pair Isabel and Jasper were busy puzzling (see figure 2). During this puzzling, it seemed again that folding helps with finding equal lengths of sides, naming angles, and discovering similarly shaped objects.

Alternately, Isabel or Jasper took the initiative. After 10 minutes Jasper seemed to had found a formula for sin(2α). When he tried to explain it to Isabel, he started to doubt again. But then Isabel also managed to produce the reasoning and they were confident and euphoric about their result. They also started working on cos(2α) and within 5 minutes they had found it, too. They really seemed very satisfied with this successful experience. Especially when it turned out that they had found the 'right' formula.

When the 20 minutes were up, the teacher took the floor again centrally. A group got to present their discovery with the homeschoolers online included. Everyone could follow along and relate the discovery to their own strategy.

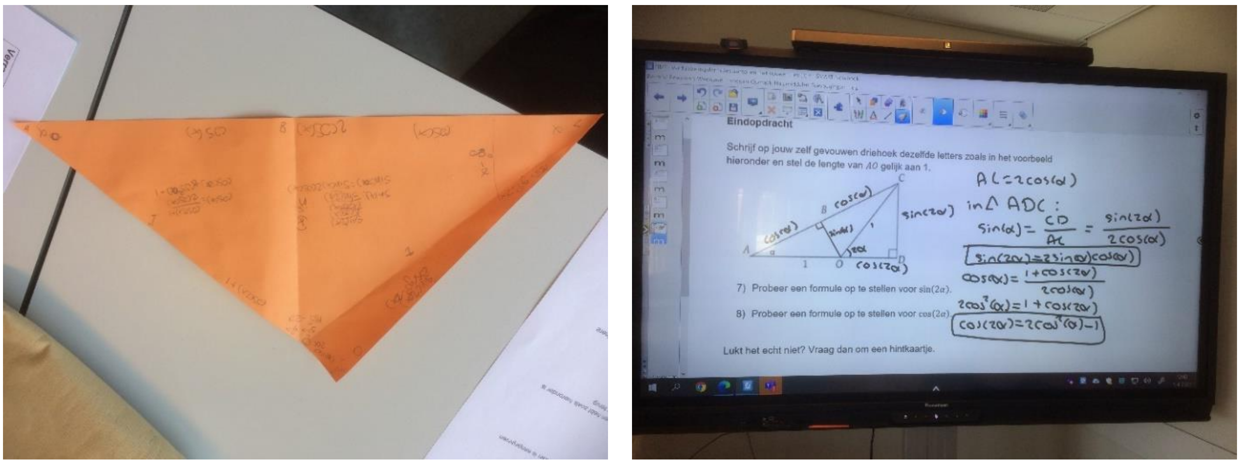

Figure 3. Unfolding the formula

Reflection

For many students, the transition from $\sin(x) = \frac{\sin(2x)}{2\cos(x)}$ to the formula for sin(2x) was not obvious, but we feel that this lesson succeeded in engaging students in the discovery of the angle doubling formula. They almost all actively participated in the activities. Will this lesson influence their confidence and creativity in mathematical inquiry and their memorization of the formulas? Time will tell.

Afterwards, we asked students for their opinions. They thought it was a fun lesson, only that it took a long time for such a small result. Apparently, we have not yet convinced all students that the journey is certainly as important as the final destination ...

Apricot jam

The graph of a function and what can be observed from it

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Identifying the problem and learning goals

In high school, we devote quite a few hours to building and understanding the concept of function. Although we try and try to bring the concept of function closer to students in different ways, with tables, graphs, formulas, etc., we notice that quite a few students have difficulties understanding the basic concepts. For example, the students confuse the zeros of a function and its initial value. Instead of the interval at which the function is increasing, the students consider the interval that ranges between the points in which the function attains the minimal and the maximal value. Also, some students have difficulties writing down the interval at which the function takes up positive or negative values.

We decided to bring the basic notions of functions closer to the students with a real-life situation. The target knowledge selected for this study lesson is to make sense of function properties such as the increasing/decreasing intervals, the sign, the zeros, the initial value, the extreme values, etc. The broader objectives are communication, development of investigative skills, data analysis, and connection of mathematics to the real situation.

Planning and creating the lesson plan

In the Spring of 2020, the fruit trees were frostbitten in Slovenia, so in this life situation, we have gotten an idea to support the students in making sense of the basic qualitative characteristics of a function. This was based on the following problem that the students were asked to solve.

During the summer holidays (2020), Jaka went to visit his grandmother who lives in a village near Murska Sobota. Jaka cries: "Grandma, I am so hungry. Can I have a piece of bread with your good homemade apricot jam?" and the Grandmother answered: "I'm sorry. I couldn't make jam this year. Spring weather is to blame for that."

Is Grandma right about the weather?

In groups of 3-4, students are asked to discuss, search for answers online, write down their findings, and report to other groups. Among the possible causes, the teacher points to the frost and directs the pupils to analyse the air temperature in March and April when it is time for apricot blossoms to flower.

In the second part of the lesson, students compare and analyse air temperature data and link this data to a function graph in a rectangular coordinate system. They record the findings and report to the other groups.

In the end, the teacher makes a summary and uses the plotted graph to connect the given task to the properties of functions: the initial value, the zeros, the sign, the increasing/decreasing intervals, etc.

Observation of the Study Lesson

We tried the scenario several times in the classroom. The first implementation took place with the original scenario. After hearing the problem, students searched online for the causes of the situation and wrote the findings in a joint document. The air temperature data were then searched for online by the students, exported to a spreadsheet, edited, and modelled with a polynomial function. The baseline, the zeros, the increasing intervals, etc. were obtained and analysed using a program which works with mathematical functions. In the end, the teacher made a summary and the students wrote a report at home.

It turned out that students spend much more time extracting and processing data than we would like. They spent two school hours searching and editing data and then another hour drawing and analysing the graph in more detail.

Since we wanted to carry out the activity within one school hour, we reformulated the scenario so that in the first part we still search online for the relevant information and record the causes of the situation, but in the second part of the lesson, students are given air temperature tables and a plot chart. In the modified scenario, students must analyse the data given in three different spreadsheets and link the data of one of the spreadsheets to the plotted function graph.

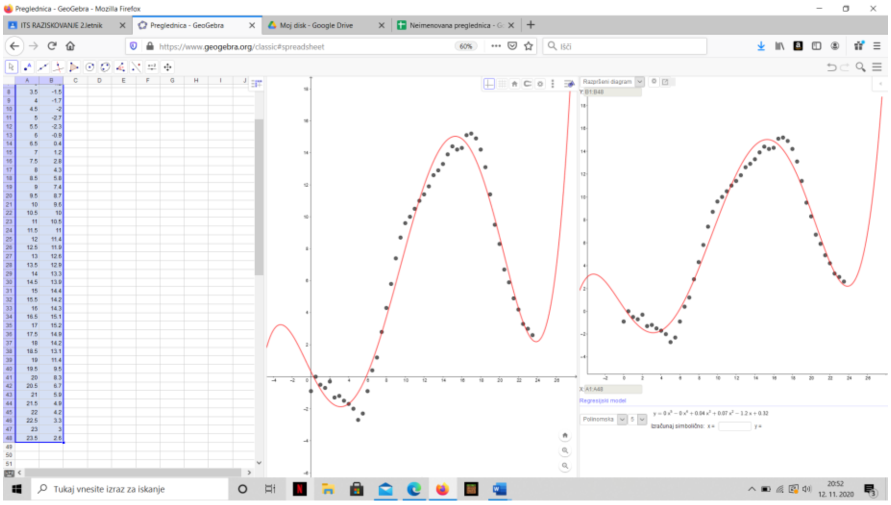

Figure 1. Data analysed with a mathematical program.

Due to the situation related to the pandemic, all follow-up activities related to the execution and accompaniment were carried out remotely via a videoconference. The teacher teaching the lesson and the observers always met 10 minutes before the start of the lesson and agreed on key steps and the manner of execution. During the lesson, observers had cameras and microphones turned off to minimize the impact on the students.

Four more implementations were done online. The students worked in groups, wrote possible answers to the question asked in a joint document, and correctly linked the spreadsheet data to the plotted graph in all four implementations.

In the first and second implementations, the students still spent too much time finding and recording the causes of the situation. During the reflection after the lessons, we concluded that this time should be shortened and more time should be dedicated to the characteristics of the function and the analysis of the graph.

MURSKA SOBOTA - RAKIČAN lon=16.1913 lat=46.6521 viš=187m	min. T [°C]
2020-03-25 05:00	-0.6
2020-03-26 05:00	0.7
2020-03-27 05:00	6.4
2020-03-28 05:00	2.9
2020-03-29 05:00	6.2
2020-03-30 05:00	4.2
2020-03-31 05:00	-5.2
2020-04-01 05:00	-2.2
2020-04-02 05:00	-6.5
2020-04-03 05:00	-3.8
2020-04-04 05:00	3.7
2020-04-05 05:00	1.8
2020-04-06 05:00	1.4
2020-04-07 05:00	-2
2020-04-08 05:00	-2.6
2020-04-09 05:00	-0.9
2020-04-10 05:00	6.5
2020-04-11 05:00	5
2020-04-12 05:00	3.5
2020-04-13 05:00	4
2020-04-14 05:00	1.4
2020-04-15 05:00	-3.1
2020-04-16 05:00	-1
2020-04-17 05:00	5.5
2020-04-18 05:00	9.5
2020-04-19 05:00	12.6
2020-04-20 05:00	8.3
2020-04-21 05:00	3.7
2020-04-22 05:00	1.5
2020-04-23 05:00	1.6
2020-04-24 05:00	0.8
2020-04-25 05:00	5.5

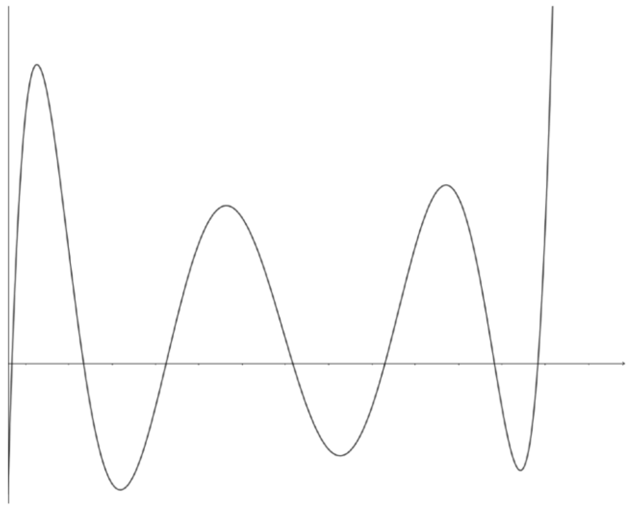

Figure 2. The data and the graph of the function corresponding to the given data

In the third implementation, the teacher reduced the time devoted to writing the answers to the question asked. More specifically, she wrote the instructions for work in a joint document. This has helped pupils to always check what needs to be done while they work. In this class implementation, after group work, the teacher orally analysed the characteristics of the functions with the students. They ran out of time and did not manage to introduce and institutionalize the function properties in more detail.

Figure 3. The answers to the main question, Jamboard

The fourth implementation of the lesson took place after minor adjustments to the revised version of the lesson plan. The class was also attended by external members of the TIME project from Gimnazija Jesenice, the Institute of the Republic of Slovenia for

Education, and the University of Ljubljana. The students started work in a similar way to other performances. Compared to the previous three versions, they spent more time observing the tables. The final part was extended by 10 minutes and in this time, together with the students, the teacher managed to make a slightly more detailed analysis of the properties of the function presented with a table and graph. Institutionalisation has not been carried out.

Temperatura zraka, Murska Sobota-Rakičan, podatki samodejnih meteoroloških postaj

Datum/ura	min. T [°C] na 2 m	Datum/ura	min. T [°C] na 2 m	Datum/ura	min. T [°C] na 2 m
2020-04-02 0:00	-3.5	2020-04-02 8:30	1.7	2020-04-02 17:00	11
2020-04-02 0:30	−4.0	2020-04-02 9:00	3.1	2020-04-02 17:30	10.6
2020-04-02 1:00	-4.2	2020-04-02 9:30	4.6	2020-04-02 18:00	9.8
2020-04-02 1:30	-4.6	2020-04-02 10:00	6.8	2020-04-02 18:30	8.3
2020-04-02 2:00	-5.1	2020-04-02 10:30	8.0	2020-04-02 19:00	7.0
2020-04-02 2:30	-5.3	2020-04-02 11:00	8.8	2020-04-02 19:30	5.8
2020-04-02 3:00	-5.4	2020-04-02 11:30	8.9	2020-04-02 20:00	4.2
2020-04-02 3:30	−6.0	2020-04-02 12:00	9.3	2020-04-02 20:30	2.9
2020-04-02 4:00	-5.9	2020-04-02 12:30	9.8	2020-04-02 21:00	2.4
2020-04-02 4:30	-6.3	2020-04-02 13:00	10	2020-04-02 21:30	1.1
2020-04-02 5:00	-6.5	2020-04-02 13:30	10.3	2020-04-02 22:00	0.0
2020-04-02 5:30	-6.9	2020-04-02 14:00	10.8	2020-04-02 22:30	-0.3
2020-04-02 6:00	-7.1	2020-04-02 14:30	10.7	2020-04-02 23:00	-0.5
2020-04-02 6:30	-5.5	2020-04-02 15:00	11	2020-04-02 23:30	-1.1
2020-04-02 7:00	-3.7	2020-04-02 15:30	11.3	2020-04-02 24:00	−1.5
2020-04-02 7:30	-1.3	2020-04-02 16:00	11.2		
2020-04-02 8:00	0.0	2020-04-02 16:30	11.1		

Vir: ARSO

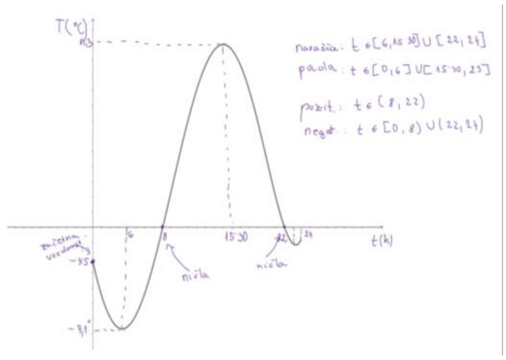

Figure 4. The data and the analysis of the function graph corresponding to the given data

The final (sixth) implementation of the lesson took place in a classroom, after minor adaptations of the scenario. After posing the problem and storming ideas about possible answers to the question asked, students were directed to an article on frostbite on trees. As in other lessons, each group was given three tables. The students first analysed the data on average and minimum air temperatures for March and April, then they were given a spreadsheet with data of varying air temperature for one day and drew the data in a coordinate plane, interpolating a graph of the function that fits the data.

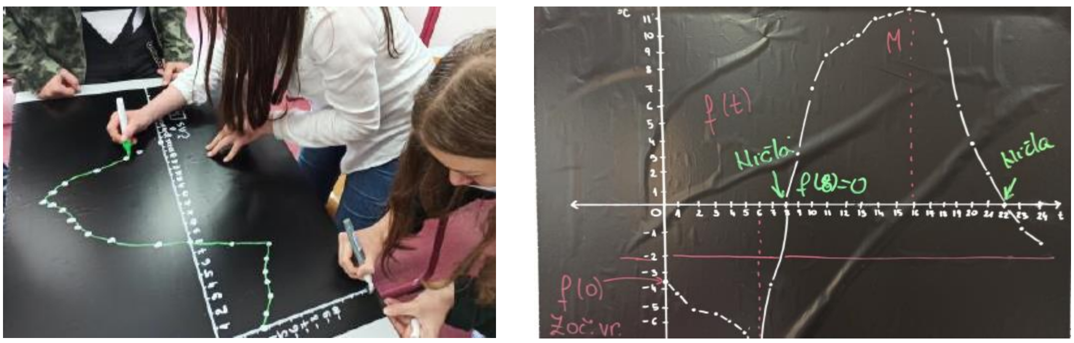

Figure 5. Drawing and analysing a function graph corresponding to the data on April 2, 2020

Reflection

At first, we set the problem too wide. In finding and analysing data, students focused more on using technology rather than reasoning and understanding the properties of functions. Therefore, the part that was used in the scenario to search for air temperature data was replaced by three given tables and a plotted graph of the function corresponding to the temperature data of one of the tables. Since, despite the change, the institutionalization has not been done in any implementation, we recommend:

1. to allocate more time for the execution of the scenario (e. g. two school hours),
2. to divide the part intended to analyse the tables and link to the function graph into two parts; in the first part, students link the table to the graph of the function and, together with the teacher, discuss and record the definitions and properties of functions; and in the second part, the students write down and give arguments for their solution to the problem,
3. if the students draw the graph themselves, each group should obtain a spreadsheet with different data (including data for a few consecutive days), thus illustrating the continuation of the function graph in the aggregate analysis.

After the reflection on the lesson, on which the external members of the TIME project were also present, it can be summarized that the set activity based on the real situation lays good foundations for students to understand the basic concepts related to functions. We have seen that the activity itself also indirectly develops teamwork, research skills and communication between students.

The attic room

Investigating geometric patterns to calculate sums

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Gimnazija Franc Miklošič Ljutomer, Ljutomer, Slovenia

Identifying the problem and target knowledge

In our school, we teach the sum of the first n positive integers and the sum of the first n odd positive integers in the final year of high school when we study sequences. In the first three years, we mention these two problems a few times to the more successful students, because they can use these facts at math competitions for solving different problems. Even those students forget both formulas, so we have decided to teach these two formulas to all first-grade students in such a way that they could recall the formulas easier. The main idea is to teach the formulas in connection to real-life problems and by seeing them from a different perspective, with the geometric visualization of the solution.

The target knowledge for the lesson are thus the two formulas

$$1 + 2 + 3 + \ldots + n = \frac{n(n+1)}{2}$$

$$1 + 3 + 5 + \ldots + (2n - 1) = n^2$$

and the broader goals are development of inquiry skills, investigation of patterns, discovery of connections between geometry, numbers and algebra, development of mathematical thinking, use of mathematical language and symbols and justification of the results of own inquiry.

Planning and creating the lesson plan

The idea for the situation itself was born while arranging the attic. Since the ceiling in the attic room is not horizontal, this presents many opportunities for arranging various elements on the wall, be it furniture or paintings. If we want to use standard elements, which are usually rectangular, we are faced with the use of mathematics in one way or another. The situation we have conceived is a broader problem that can be solved with different approaches and offers different possibilities for exploration. If we regulate the situation with the dimensions of the wall, we can ask different questions. Below we will introduce one of the many options for exploration. We could certainly place the situation in different contexts, thus creating new opportunities for students to explore and learn different mathematical laws.

The problem that the students dealt with and we present in this report, has developed over a long period of time. We searched for the appropriate text, questions, dimensions, etc. We called it simply the attic room. The text of the task is:

> *Janko and Metka are arranging the attic room. On the grey parts of the triangular shape of one wall (see the picture) they will hang framed photographs measuring* 20 cm x 20 cm*.*
>
> *Explore the wall tiling with the maximal number of photos without overlap.*

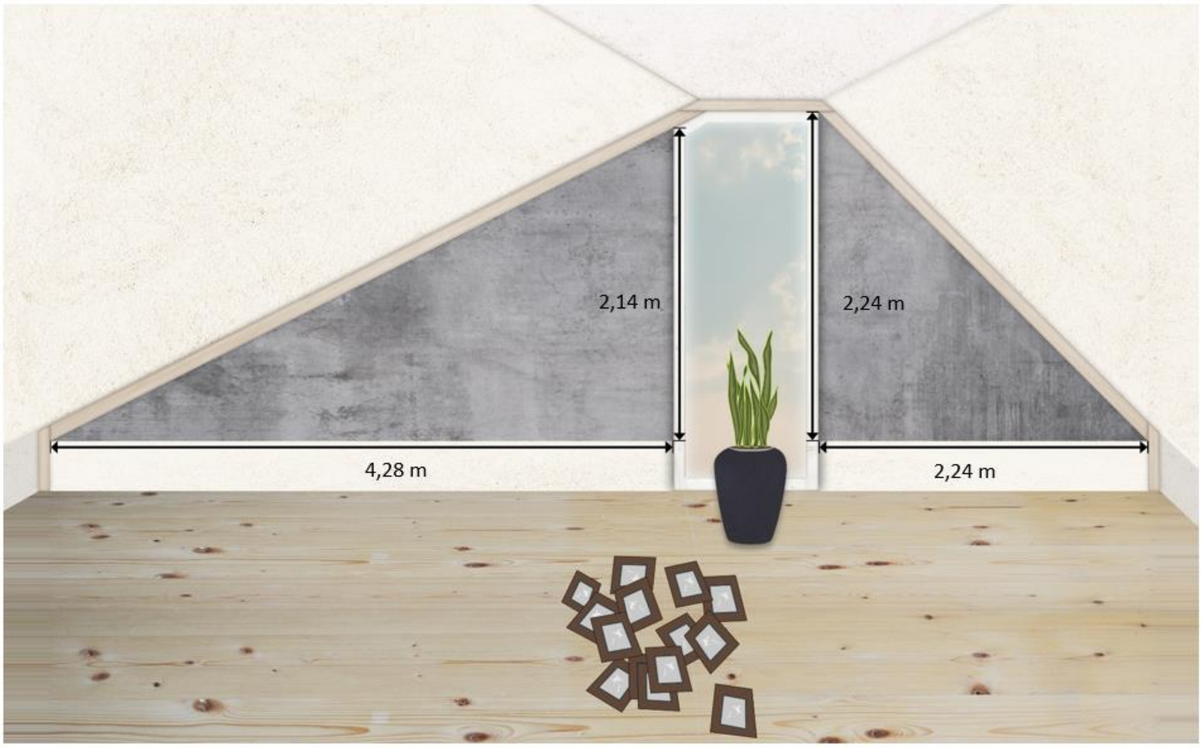

Figure 1. The two attic walls in the shape of a right-angled triangle

When designing the situation and planning the course of the work, we decided to direct the students to explore the wall tiling with the maximum number of photos without overlapping. We planned 2 school hours to implement the lesson, which is 90 minutes.

We planned the implementation in several parts. In the introduction, the teacher introduces the problem and invites the students to reflect and write down ideas of what they could explore at all. After reviewing these ideas, students are directed to explore the possible arrangements of the photos on the wall so that they cover as much of the wall as possible. Students must keep in mind that framed images cannot be cut.

Students explored in groups or pairs. They wrote down their ideas on an ongoing basis and in the meantime also presented them to other groups or pairs. When a concrete answer was found, the students focused on finding a general rule or formula. This has also been proved mathematically.

Observation of the Study Lesson

In the first devolution, the teacher presented the problem and asked the students to write briefly what they could observe and research. Here are some ideas from the students:

> *How many photos can we hang on each wall?*
> *How many more photos are on the larger, compared to the smaller wall?*
> *What part of the wall will be left empty?*
> *What is the wall area tiled by the photos? What is the wall area that is left empty?*
> *What is the hypotenuse of triangles?*
> *What are the sizes of the angles of the triangles?*

After presenting the ideas, they decide to explore how many photos they can hang on the wall. In the second devolution, the teacher asks the students to explore how many more photos we can hang on the left than on the right wall if we don't cut the photos. Most of the work was done by calculating the area of a triangle and dividing that area by the area of one photo.

For the left wall:

$$\frac{(428\ cm \cdot 214\ cm)}{2} : (20\ cm \cdot 20\ cm) = 45796 : 400 = 114{,}96.$$

For the right wall:

$$\frac{(224\ cm \cdot 224\ cm)}{2} : (20\ cm \cdot 20\ cm) = 25088 : 400 = 62{,}72.$$

The students found that we can hang a maximum of 114 photos on the left wall and 62 on the right, which is of course too much if we assume that we do not cut the framed photos. Others decided to work by drawing a sketch of the wall and drawing squares representing the photos. Students which did not draw the sketches of the wall to scale mostly counted the squares and thus got an approximate number of photos. They did not notice the rule, as in some places, the number of photos in a row increased by one, and then again by two or even three. Students who drew a sketch of the wall to scale found that on the left wall, in each subsequent row, the number of squares increased by two and on the right by one.

For the left wall: $1 + 3 + 5 + 7 + 9 + 11 + 13 + 15 + 17 + 19 = 100.$

For the right wall: $1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 + 10 = 55.$

In the third devolution, the teacher asks the students to write down a formula that could be used to calculate the number of pictures if they had n instead of 10 rows of pictures. Most of the students started work on the right wall first.

Some helped themselves with the sum

$$(1 + 10) + (2 + 9) + (3 + 8) + (4 + 7) + (5 + 6) = 5 \cdot 11 = \frac{10}{2} \cdot 11 = \frac{10 \cdot 11}{2}$$

and then concluded that for n lines the formula is equal to $\frac{n(n+1)}{2}$.

Others completed the sketch of the images to a rectangle and wrote down the formulas:

$$1 + 3 + 5 + 7 + 9 + 11 + 13 + 15 + 17 + 19 = \; 100 = 10^2$$

$$1 + 3 + 5 + 7 + 9 + \ldots + (2n - 1) = n^2$$

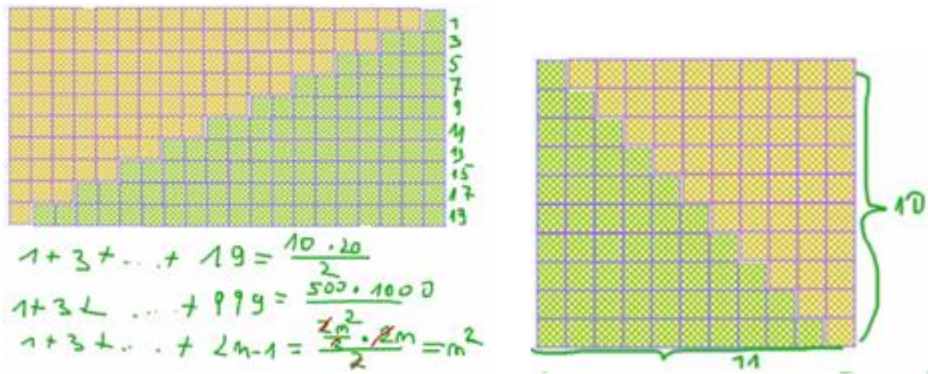

Figure 2. Completing the walls into a rectangle.

Since none of the students derived a formula for the sum of the first n odd numbers, they finally showed an analytical and geometric derivation together with the teacher and wrote down the form.

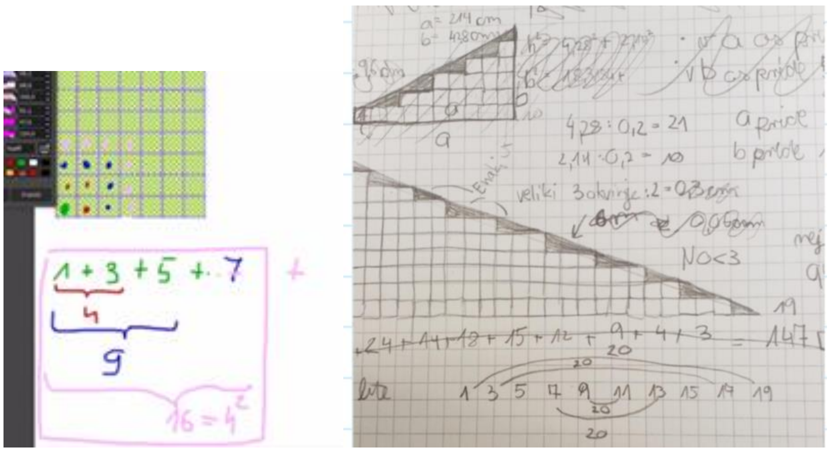

Figure 3. Possible ways to explain the formula for the sum of odd numbers using geometry.

Reflection

The lesson was clearly structured and the teacher guided the students in their work. There was plenty of time to execute the lesson plan. In the initial part, students gave different ideas on how to hang pictures on the wall, upright, obliquely, one over the other, etc. The teacher directed the students to explore hanging pictures next to each other, so that the pictures stand upright and do not overlap.

Students explored in a variety of ways, most of them by drawing, one by calculating the angle of inclination, one by calculating the area of the wall and squares, etc.

There were several problems with the notation in general and this was the aspect with which the teacher helped and guided quite a bit.

In future implementation, we would recommend instructing students to draw a sketch at the appropriate scale and to give students plenty of time to derive and write down the formula for the general case.

Curved roof

Modelling and calculating measures of geometrical shapes

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Defining the problem and target goals

Students are always interested in the usefulness of what they have learned. In subjects such as physics, chemistry, and biology, it is not difficult to find examples from everyday life, while in mathematics it is more challenging to find a meaningful real-life example that also covers the knowledge to be taught.

As part of the school's cooperation with the local environment, we were given an interesting task by an entrepreneur who wanted to see what kind of solution our students would come up with. In exchange for cooperation, he offered a tour of the company and its presentation. In this way, the students got the opportunity to apply their knowledge in practice and to be rewarded for it. With this activity, our goal was to increase interest and motivation in learning mathematics itself, and with that, we also hoped that the students would be able to apply the obtained knowledge to concrete examples and not just theoretical ones.

Planning and creating a lesson plan

The duration of the lesson is two school hours (90 minutes). Students work in groups, 4-5 students per group, and are expected to solve the given tasks, participate in the discussions, present their conclusions, and compare various solutions.

To start the lesson, the teacher starts with the introduction of the topic:

> *You certainly have not thought about your house yet, nor have you thought about what kind of a roof it will have, but you may find that this is an interesting problem and that the skills to solve it might be important to you in the future.*
>
> *You all know the usual shape of the roof, where the ridge is triangular. Today we will take a closer look at the curved roofs. In addition to being an interesting shape, such roofs also have practical advantages. Can you imagine what happens to the snow on*

such a roof, how the rain drains away, how it behaves in severe winds, and how much extra space you gain?

The teacher lets the students have a short discussion, motivating them to talk to each other about the two options for the roof.

The problem that the students should solve has three questions, all concerned with the same situation:

Consider a warehouse, like the one in the picture. It is 15 m long, and 12 m wide. The entrance is 7.5 m high, while the height of the whole building (from top to bottom) is 9,5 m.

The roof is made out of a rectangular piece of material which is curved and placed on the warehouse. To reinforce the roof, we use five pillars (blue in the picture). The pillars are evenly distributed along the width of the entrance. Our goal is to see how much material we need to use to produce such a roof and how much space we gain by having it.

Figure 1.

Warehouse with a curved roof

Model the roof using the mathematics you already know and calculate:

- *the size of the rectangular area that we will bend into a curved roof,*
- *the size of the support pillars,*
- *the difference between the volumes of the warehouses with triangular and curved roofs.*

In addition, the students are asked to provide general formulas that could be entered in Excel to answer the above questions automatically for different sizes of the warehouse.

The students should divide the tasks among themselves within their group and after the discussion, someone should write down the solutions on the prepared paper. Students will report their strategies to the whole class and compare the answers between the groups.

Observation of the Study Lesson

At the beginning of the lesson, the teacher handed out the materials: worksheets with presented problems and blank presentation sheets.

As they started thinking about the problem and determining which quantities to use, a few problems arose. The students questioned how to get the length of the rectangular piece of material from the data that is given in the task. The tasks they solved were posted on the Padlet in real-time, so it was possible to see how the groups were progressing and where they needed some help from the teacher.

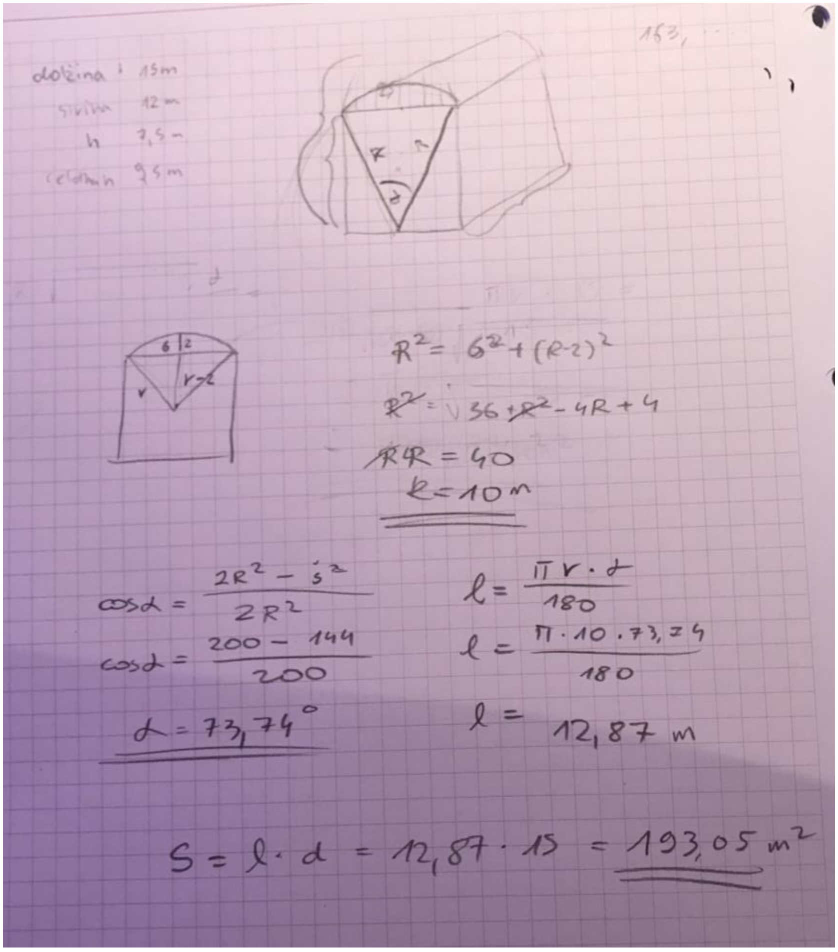

Figure 2. Students' work – modelling the roof as a circular arc and calculating the angle, length and area of a circular segment

Students mostly solved the three tasks in the order in which they are written. Some, but not all, groups decided and distributed the tasks, solving several questions at the same time. These groups were more successful. Most students chose to model the roof with a circular arc (part of a circle), and some chose to work with an ellipse. In that case, they had to find the angle to calculate the length of the arc using trigonometry and in the second task, they use the quadratic equation to calculate the length of the pillars. The final task, calculating the volume, was not so difficult because the warehouse has the shape of a prism and the base of that prism is the cross-section that we see in the picture.

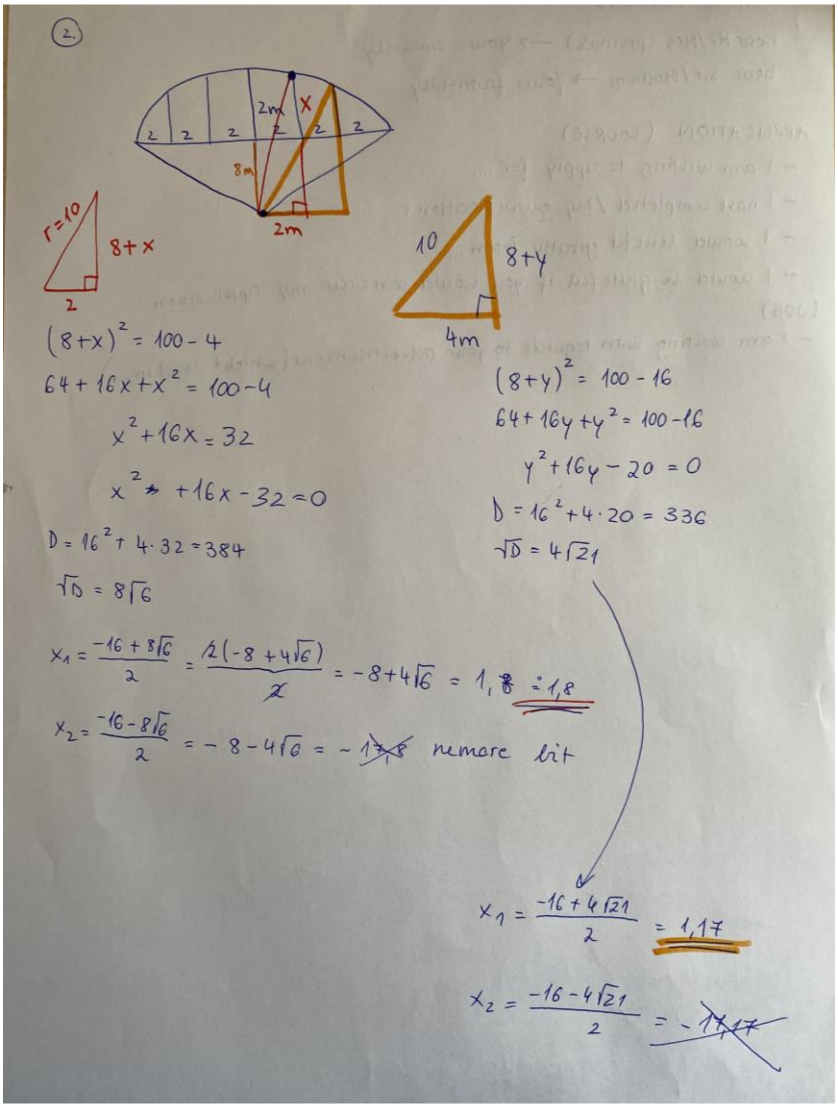

Figure 3. Students' work – calculating the length of the pillars using the formula for the circle

As expected, the general formula and writing the general formula in Excel turned out to be the most difficult part. It might be also because the lessons were done online and students needed more time to communicate their ideas.

At the end of the lesson, we asked the students for their feedback and were very happy that it was positive. Here is one of many similar answers:

> *I personally liked the lesson as it was something a little bit special. I especially liked working in small groups, where we communicate with each other (as in a classroom) and try to come up with a solution together. I also found it interesting to see how our "geekiness" about geometric bodies can be useful in everyday life and not just in a math notebook. I would gladly repeat such a lesson.*

Reflection

The lesson was held four times in classes for third- and fourth-year students (for third-year students, geometry is part of their regular curriculum, while for fourth-year students the lesson was conducted as a repetition for their final exam). This lesson was very structured, but led only partially by the teacher. The students were engaged in many different aspects of the lesson: writing ideas, calculating, discussing, and presenting conclusions on the Padlet and in front of their classmates.

Based on our experience we would recommend:

- to conduct the lesson in the computer classroom or to prepare at least one school laptop per group, or to ask the students to bring their laptops with them,
- to find out in advance how well students use Excel.

The birthday paradox

Mathematical explanation of counterintuitive probability facts

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Defining the problem and target goals

We have chosen a task that is not new, but it is interesting because of its final result. Usually, when students are given a new task, they think about the task and look for quick ways to solve it. In doing so, they usually first rely on conventional procedures, mostly solvable with proportions; when you increase (decrease) one quantity, the other also increases (decreases). Through this task and knowledge of the basics of probability, the students realize that the result is far from expected.

In this study, we have chosen a task where logic fails, and the final result, which is obtained through mathematical calculations, causes a wow effect for all students who have not encountered the task before.

In this way, we wanted that the students think about the tasks not only through logic, but also to support the results with the calculations themselves, and secondly, that they see the usefulness of what they have learned.

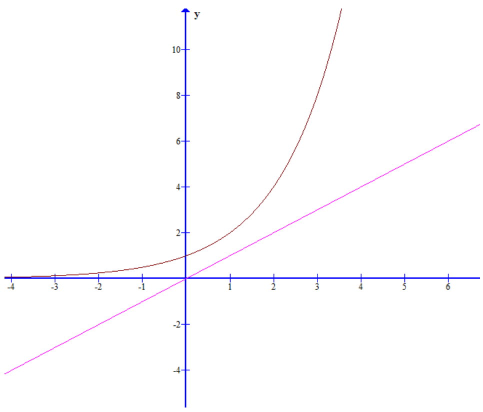

Figure 1. Linear vs. exponential growth

Students should be reminded that it is much easier to think about the logic of solutions to problems that are linear than to solve tasks where quantities change exponentially, because the brain has a hard time mastering the high speed of the increase or decrease.

For example – when the students are asked a simple question: "*What's the chance of getting 10 heads in a row when flipping coins?*" they never hesitate with an answer.

"*Getting one head is a 50 % chance, getting two heads is a 25 % chance, so getting ten heads is 10 times harder – 50 % divided by 10 is a 5 % chance.*"

And our wish was to avoid such answers in the future with the help of understanding the following task:

> *How many people do you think it would take on average, to find two people who share the same birthday? How many people do you need in the room to have a 50 % chance?*

In this task intuition causes problems, but math "saves the day".

Planning and creating a lesson

The duration of the lesson is one hour (60 minutes). Students work in groups, 4-5 students per group. The lesson is organized into two parts. Students are expected to solve mathematical tasks, participate in discussions, and compare, define, and present their conclusions.

First part: practical work

They must answer the following questions:

How many people (papers with birthdays) do you need, to have at least two people with the same birthday? Why do you think so? How many people do you need, to have at least a 50 % chance that two people will have the same birthday? For each in the group write down his/her prediction. What is the average in the group? Students write their findings on the board and explain them.

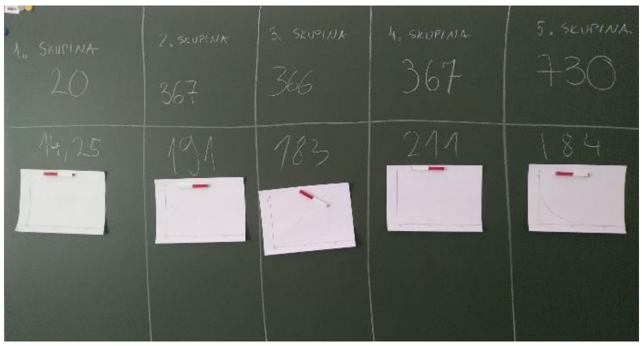

Figure 2. Students' answers for the 100% and 50 % chance

After the first report, students write 5 different birthdays of their friends on five slips of paper. They fold the papers and put them in a bag. The teacher shuffles the collected slips, and one student from the group then randomly selects 5 and then another 20 slips.

Then they randomly select 23 papers from the bag and check if any date is repeated on any of the written papers. They write down the findings and discuss if they were surprised by the result and why. In this way, the students from each group recreate the situation on their concrete example (they have birthday dates written on pieces of paper) and come to a solution by counting, and communicating within the group, without calculating. They write down their findings and present them to others.

Second part: calculations

How would you calculate the probability that there are two people in a group of five who were born on the same date? To simplify the calculation, we assume there are no leap years (we exclude 29th February), assume the other 365 days are equally likely and that birthdates are independent.

What is the probability that in a group of 23 people two people have a birthday on the same day? What do you think about the calculated probability? Did the number surprise you? Why?

There was no match for 5 slips, which did not surprise the students. When counting up to 20 or 25 slips, some groups found a match. These groups were surprised by the result, as they did not expect a match.

Observation of the Study Lesson

At the beginning of the lesson, the teacher handed out the materials and explained the task. The students wrote down birthdays. When guessing without calculation, most groups concluded that for a 100% match, we need as many different dates as there are days in the year, and for that to be certain, we would rather have a few more dates. For a 50 % match, the result was simply divided by two.

The task was solved in four classes, in a total of 17 groups. Only in two groups, they reduced the number of dates in their answer. In one, they were gambling, and in the other, they concluded that there must be some kind of a trick, otherwise they would not have gotten the task.

In the second part, where the task was repeated but solved computationally, some students had a problem starting the calculation. They calculated directly the probability that the two dates match. They soon realized that the higher the number of guests, the more options there are to check, which made their calculations very difficult. If the students themselves did not realize that it is better to calculate with the opposite event, we helped and partially guided them to this point.

The calculation then immediately followed:

$$P(A) = 1 - P(A^c) = 1 - \frac{365 \cdot 364 \cdot 363 \cdot 362 \cdot 361}{365^5} = 1 - 0{,}97286 = 0{,}02714$$

$$\Rightarrow P(A) = 2{,}71\ \%$$

For $n = 23$, we have the following calculation:

$$P(A) = 1 - \frac{365 \cdot 364 \cdot \cdots \cdot 343}{365^{23}} = 1 - \frac{365!}{342!\,365^{23}} = 1 - 0{,}4927 = 0{,}5073 = 50{,}73\ \%$$

One should note that the first form of the fraction is more suitable for using the calculator. The students will encounter problems if they try to calculate 365!.

In the end, the students in groups presented their solutions and wrote them on the board. If there was still some time left, we looked at the table of the probabilities that there is a match and the graph of the function

$$P(n) = 1 - \frac{365!}{(365-n)!\,365^n}$$

and discussed it with the students.

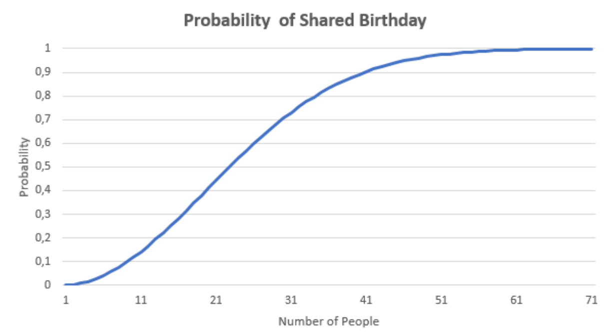

Figure 3. The graph for the probability that there is a shared birthday in a group of n people

Reflection

The lesson was held four times in classes of four graders because our curriculum foresees the probability in the last year of schooling. Based on the observation minor changes were made in the scenario. This lesson was structured and led by the teacher, but the students were engaged during the entire lesson: writing dates, counting, calculating, discussing, and presenting conclusions.

There were exactly 23 students in the first class, where the scenario was carried out (the number of people that the probability of matching is 50 %). This number can be changed, and the lesson gets upgraded, as the students have to additionally find out (calculate) the right number of people to have 50 % for a match.

Further suggestions based on previous experience are to:

- allocate more time for the discussion at the end, when you talk with the students about the calculations made in the table (the probability of a match based on the number of people in the room) and the graph, the shape of the graph, the growth rate of the pairs you are comparing (depending on the size of the group).
- for the presentation, choose groups that solved things differently (it is not necessary for all groups to report, at the end just encourage them to comment on the results and say if they noticed or calculated anything additional).